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# A Modal Combination Program for Dynamic Analysis of Structures

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# *A Modal Combination Program for Dynamic Analysis of Structures*

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## ABSTRACT

A method of determining the response of composite structures subjected to a sinusoidal forcing function is developed. The method uses characteristic rigid body and elastic deflected shapes of the components. The input required and limitations of a program using the method are stated. A sample problem is used to compare results with other methods.

## I. INTRODUCTION

The modal combination program for dynamic analysis of structures, as presented in this Report, determines the response of a composite linear structure subjected to low-frequency sinusoidal base motion of a restrained structure or low-frequency sinusoidal forces at points of a free structure. The program is based on a method described in JPL TR 32-330 by Walter Hurty. The intention in developing the program was primarily to determine the undamped modes of a composite structure and secondarily to get response to sinusoidal forcing functions, which was required for problems related to current testing practices and closed-loop stability of autopilot-controlled space vehicles.

Models of components (basic systems) in forms of geometry, normal modes, frequencies, lumped masses, and elastic properties are required. Systems are developed from basic systems when the required compatibility with the composite is imposed. Operation is divided into five parts: (1) basic system processing, (2) system processing, (3) composite processing, (4) forced response calculation, and (5) point acceleration and member stress calculation. Any adjacent parts of the program may be used in a single computer run.

The following calculations are performed in basic system processing: (1) geometry, member properties, normal mode shapes, frequencies, and modal damping coefficients are read in; (2) rigid body modes, modes describing the independent motion of redundant supports (constraint modes), modes associated with concentrated loads at unrestrained points (attachment modes), and associated reactions are calculated, and (3) the modal matrix, mass matrix, stiffness matrix, and damping matrix are formed.

The following calculations are performed in system processing: (1) required compatibility is imposed, and (2) transformations from composite coordinates to system coordinates, mass, stiffness, and damping matrices of composite are developed.

The following calculations are performed in composite processing: (1) undamped eigenvalues and eigenvectors, and damped eigenvalues and eigenvectors are found, (2) the transformation from uncoupled coordinates to composite coordinates and the uncoupled combined mass and damping matrix are developed, and (3) point accelerations of undamped mode shapes are punched by the computer if desired.

The following calculations are performed during response calculation: (1) the generalized forcing function matrix is formed, (2) response of given control points is calculated and plotted, and (3) composite system generalized displacements for frequencies which have the largest response are punched by the computer.

The following calculations are performed in point acceleration and stress calculation: (1) point accelerations are calculated from composite system generalized displacements punched on cards and transformations saved on tape, (2) mass acceleration "forces," the associated static displacements and related accelerations, are calculated, and (3) member loads are found using the deflections associated with either modal accelerations or inertial loading.

Ingenuity is required in the use of the program primarily in defining realistic idealizations of the components.

Future extensions of the program will allow non-sinusoidal forcing functions, as most of the program is not limited by this restriction.

## II. DEVELOPMENT OF METHOD FOR BASIC SYSTEM PROCESSING

Any structural unit, within the limitations of the program, may be used as a basic system.

The set of equilibrium equations for each degree of freedom of a basic system in matrix notation is:

$$[m]\ \{\ddot{u}\} + [c]\ \{\dot{u}\} + [k]\ \{u\} = \{f\}$$

$[m]$, $[c]$, $[k]$ are the mass, damping, and stiffness matrices. $\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ and $\{f\}$ are the displacement, velocity, acceleration and loading vectors. The elements of the loading and displacement vectors are assumed to have the form $A\ e^{j\omega t}$. Future extensions of the modal combination program may provide for other types of loading.

Any linear combination of point displacements ($\{u\}_i$) may be associated with a generalized displacement ($P_i$). The coefficients of the generalized displacement (displacement when $P_i$ is unity) will be called a modal vector ($\{\phi\}_i$). Some particular modal vectors have useful properties that will be mentioned later. An array of modal vectors will be called the modal matrix ($[\phi]$). A modal vector times the associated generalized displacement is the displacement of the points describing the generalized displacement. Therefore, the modal matrix times the associated matrix of generalized displacement is the matrix of displacements of points corresponding to the given values of the generalized displacements ($\{u\} = [\phi]\ \{P\}$).

Using the modal matrix $[\phi]$ as a transformation, $[M]$, $[C]$, $[K]$ and $\{F\}$ are defined as follows:

$$[M] = [\phi]^T\ [m]\ [\phi]$$
$$[C] = [\phi]^T\ [c]\ [\phi]$$
$$[K] = [\phi]^T\ [k]\ [\phi]$$
$$\{F\} = [\phi]^T\ \{f\}$$

Therefore:

$$[M]\ \{\ddot{P}\} + [C]\ \{\dot{P}\} + [M]\ \{P\} = \{F\}$$

where $P_i$ is the participation of the $i^{th}$ mode.

Rigid body modes $[\phi_R]_0$ are the displacements of the points in the basic system when there is a unit displacement (either translation or rotation) of the coordinate axis from which the points are described. These modes are calculated from geometry only. There are six such modes if the structure is stable as a free body (no internal hinges).

Normal modes uncouple the equations of equilibrium when used as a transformation of coordinates. Normal modes are input to the program on IBM cards and can be determined either experimentally or analytically.

Constraint modes are those modes which result from prescribing unit displacements at redundant restraints.

If it is desired to allow relative displacements between the restraints, constraint modes are used.

Attachment modes are those modes which result from a concentrated load at a point. If another system is attached at an unrestrained point of a system and it is felt that the resulting imposed loads will have a significant effect on mode shape or stresses, attachment modes are used.

Since idealized structures may be used which are not stable with a restraint removed but which are not completely described without independent support motions, the concept of links is introduced. This allows the calculation of the required modes by geometry alone without recourse to elastic properties. Links are portions of the structure that are hinged to the main body. The main body has six rigid body modes, and each link adds at least one additional rigid body mode to the system. These additional rigid body motions of the link are superimposed on the rigid body motion of the basic system as a whole.

The displacement of a point on the link is 0 for the degrees of freedom defining link motion. The hinge line (hinge point if there are two additional independent generalized displacements) is defined by the displacements at these points.

$$\{u\}_L = [\phi]_L \{P\}_L = \{0\}$$

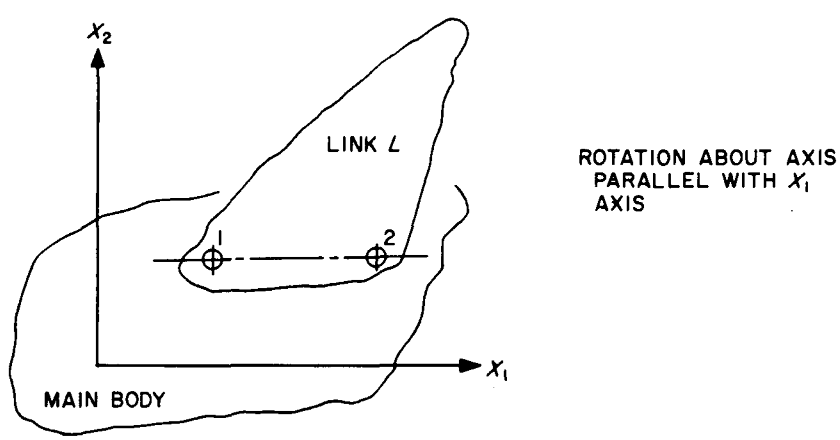
The matrix $[\phi]_L$ is extracted from the rows of the rigid body modal matrix of the main body $[\phi_R]_0$ corresponding to the degrees of freedom defining the motion of link "$L$." This implies only that the origin of coordinates of points on the link is the same as points on the main body. The elements of the matrix $\{P\}_L$ are the rigid body generalized displacements of link "$L$." There are precisely as many dependent generalized displacements as there are degrees of freedom defining joint motion. Partitioning $[\phi]_L$ and $\{P\}_L$ into dependent and independent generalized displacements results in the following equations.

$$\{u\} = \left[ \phi_I \mid \phi_D \right] \left\{ \begin{array}{c} I \\ \hline D \end{array} \right\} = \{0\}$$

$$\{D\} = -[\phi_D]^{-1} [\phi_I] \{I\}$$

$$\{P\} = \left[ \begin{array}{c} 1 \\ \hline -[\phi_D]^{-1} [\phi_I] \end{array} \right] \{I\}$$

In order to invert the matrix $[\phi_D]$, it must be square and non-singular. This condition is satisfied if the degrees of freedom used to define the link motion are necessary to define the common joint motion, and the number of common displacements equals the number of dependent generalized displacements for the link. For equilibrium of all parts of the structure, the generalized displacement of the link chosen to be independent must in fact be independent. For example, in Sketch No. 1 it would be improper to match the displacements along the $x_1$ axis




**Sketch No. 1**

at both point 1 and point 2 since one of these is redundant, or to match the rotation about the $x_1$ axis at either point 1 or point 2 since this would prevent the desired motion. It would also be improper to choose a rotation about either the $x_2$ or $x_3$ axis or a translation along the $x_1$ axis as an independent generalized displacement of the link since these motions of the link are not independent of the main-body rigid-body motion.

The link displacements can now be written in the following form for points on the link:

$$\{u\} = \begin{bmatrix} \phi_I & \phi_D \end{bmatrix}_L \begin{bmatrix} 1 \\ \hline -[\phi_D]^{-1}[\phi_I] \end{bmatrix} \{I\}$$

$$\text{Defining } [\phi_R]_L = \begin{bmatrix} \phi_I & \phi_D \end{bmatrix}_L \begin{bmatrix} 1 \\ \hline -[\phi_D]^{-1}[\phi_I] \end{bmatrix}$$

$$\{u\}_L = [\phi_R]_L \{I\}_L$$

The matrices $[\phi_I]_L$ and $[\phi_D]_L$ are the rows of the rigid body modal matrix $[\phi_R]_0$ associated with points on the link and partitioned into independent and dependent generalized displacements. The independent generalized displacements of the links follow the rigid-body generalized displacements of the main body in the matrix of generalized displacements.

An attempt will be made to justify using a truncated set of normal modes. If a complete set of normal modes, plus rigid-body modes and modes which describe the deformation of the structure when one restraint is given a unit displacement (constraint mode), is used, the modal vector matrix is square and equilibrium is ensured in all directions at all points (see JPL TR 32-530), but the size of the problem is unchanged since the number of rigid-body modes plus constraint modes equals the number of restraints, and the number of normal modes equals the number of degrees of freedom. Modes associated with the higher frequencies have less effect on the response of the structure than modes associated with lower frequencies when the structure is excited at low frequencies. Therefore, the modal matrix is reduced by eliminating higher frequency modes, and equilibrium is no longer assured in all directions at all points, but only in the generalized displacements which have been retained. Additional justification for neglecting the higher frequency modes comes from the lower confidence that can be placed in the higher frequency modes which have been experimentally evaluated.

For test and computational simplicity, a second type of constraint mode (attachment mode) has been provided which allows concentrated loads due to the attachment of other systems at points which are not restrained in the analysis of the system under consideration. If a complete set of normal modes were used, these additional modes would be redundant as they are linear combinations of the complete set of normal modes. There is no proof given that the attachment modes are not a linear combination of the truncated set of normal modes, but due to the extreme truncation employed on large systems, this coincidence is not expected in practice. The substitution of attachment modes for constraint modes, where appropriate, allows the use of existing analysis as the basis of normal modes. This substitution allows physical testing for evaluation of mode shapes of a real structure without requiring extensive fixtures to restrain a multitude of attachment points, and also a greater part of the motion is due to normal modes for which damping can be measured in log decrement tests.

The mass, damping, stiffness and loading matrices can be partitioned, corresponding to the different types of modes used, with the order of modes in each case being rigid body modes (R), constraint modes (C), attachment modes (A), and normal modes (N).

$$\left[\begin{array}{c|c|c|c} M_{RR} & M_{RC} & M_{RA} & M_{RN} \\ \hline M_{CR} & M_{CC} & M_{CA} & M_{CN} \\ \hline M_{AR} & M_{AC} & M_{AA} & M_{AN} \\ \hline M_{NR} & M_{NC} & M_{NA} & M_{NN} \end{array}\right] \left\{\begin{array}{c} \ddot{R} \\ \hline \ddot{C} \\ \hline \ddot{A} \\ \hline \ddot{N} \end{array}\right\}$$

$$+ \left[\begin{array}{c|c|c|c} C_{RR} & 0 & 0 & 0 \\ \hline 0 & C_{CC} & 0 & 0 \\ \hline 0 & 0 & C_{AA} & 0 \\ \hline 0 & 0 & 0 & C_{NN} \end{array}\right] \left\{\begin{array}{c} \dot{R} \\ \hline \dot{C} \\ \hline \dot{A} \\ \hline \dot{N} \end{array}\right\}$$

$$+ \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & K_{CC} & 0 & 0 \\ \hline 0 & 0 & K_{AA} & K_{AN} \\ \hline 0 & 0 & K_{NA} & K_{NN} \end{array}\right] \left\{\begin{array}{c} R \\ \hline C \\ \hline A \\ \hline N \end{array}\right\} = \left\{\begin{array}{c} F_R \\ \hline F_C \\ \hline F_A \\ \hline F_N \end{array}\right\}$$

Several elements of the stiffness matrix are 0 or diagonal. The elements associated with rigid body modes $[K_{NC}]$ and $[K_{CN}]$ are shown to be 0 in JPL TR 32-530. $[K_{NN}]$ is diagonal due to the orthogonality properties of normal modes; $[K_{AC}]$ and $[K_{CA}]$ are 0 since the load in the attachment mode has no motion in the constraint modes.

The generalized mass matrix $[M]$ is defined by the relationship $[M] = \phi^T \lceil m \rfloor [\phi]$ because the point mass matrix $\lceil m \rfloor$ is diagonal. It is clear that any degree of freedom that has no motion in either mode associated with the element of $[M]$ cannot contribute to that element. For this reason, and because of symmetry, some shortcuts can be taken in the evaluation of $[M]$. Only unrestrained degrees of freedom need be considered in evaluating elements associated with normal and attachment modes. The only modification of this for constraint modes is to add the mass of the constraint degree of freedom to the diagonal elements of $[M_{CC}]$ (modal displacement is unity). The mass of the constraint degree of freedom is multiplied by the rigid-body displacement in evaluating the elements of $[M_{CR}]$. All degrees of freedom are used in evaluating $[M_{RR}]$.

The stiffness matrix $[K]$ is partitioned into degrees of freedom that are unrestrained $(U)$ degrees of freedom that are associated with constraint modes $(C)$ and restrained degrees of freedom which are omitted.

The equations of equilibrium associated with the unrestrained degrees of freedom, and the loads applied in the constraint modes can be written:

$$\left[ K_{UU} \mid K_{UC} \right] \begin{bmatrix} u_C \\ \hline 1 \end{bmatrix} = [0]$$

$$[u_C] = -\,[K_{UU}]^{-1}\,[K_{UC}]$$

$$[\phi_C] = \begin{bmatrix} -\,[K_{UU}]^{-1}\,[K_{UC}] \\ \hline 1 \end{bmatrix}$$

The rows of $[\phi_C]$ must, of course, be rearranged into their original order and the restrained degrees of freedom added. In order to invert $[K_{UU}]$, it must be square and non-singular. The calculation of $[K_{CC}]$ is as follows:

$$[K_{CC}] = [\phi_C]^T\,[K]\,[\phi_C]$$

$$= \left[ -K_{CU}\,K_{UU}^{-1} \mid 1 \right] \begin{bmatrix} K_{UU} & K_{UC} \\ \hline K_{CU} & K_{CC} \end{bmatrix} \begin{bmatrix} -K_{UU}^{-1}\,K_{UC} \\ \hline 1 \end{bmatrix}$$

$$= \left[ -K_{CU}\,K_{UU}^{-1} \mid 1 \right] \begin{bmatrix} 0 \\ \hline K_{CC} - K_{CU}\,K_{UU}^{-1}\,K_{UC} \end{bmatrix}$$

$$= [K_{CC}] - [K_{CU}]\,[K_{UU}]^{-1}\,[K_{UC}]$$

$$= [K_{CC}] + [K_{CU}]\,[u_C]$$

The equations of equilibrium associated with the unrestrained degrees of freedom and the attachment mode loads $[R_0]$ can be written:

$$[K_{UU}]\,[u_A] = [R_0]$$

$$[u_A] = [K_{UU}]^{-1}\,[R_0]$$

$$[\phi_A] = [u_A]$$

except for restrained and constraint degrees of freedom which $= 0$.

$$\begin{bmatrix} K_{AA} \\ \hline K_{AN} \end{bmatrix} = \left[ \phi_A \mid \phi_N \right]^T [K]\,[\phi_A]$$

Since $[K]\,[\phi_A] = [R_0]$

$$\begin{bmatrix} K_{AA} \\ \hline K_{AN} \end{bmatrix} = \left[ \phi_A \mid \phi_N \right]^T [R_0]$$

From the known properties of normal modes

$$\lceil K_{NN} \rfloor = \lceil \omega^2 \rfloor\,\lceil M_{NN} \rfloor$$

Due to the lack of better information, the damping matrix is assumed diagonal when the basic system is described in terms of its modes. The damping coefficients of the original set of equilibrium equations are never defined. (This lack of definition of damping coefficients places a basic limitation on the use of the mass acceleration method for finding member loads. For this reason, the option of finding member loads using modal displacements was provided.)

Each basic system must be describable in the format of one of the four types of structures used in JPL TM 33-75:

1. Three-dimensional, pin-jointed members.
2. Three-dimensional, rigid-jointed members ($I$ is same in any direction).
3. Two-dimensional, rigid-jointed members loaded in plane.
4. Two-dimensional, rigid-jointed members loaded out of plane.

*All basic systems need not be of the same structures type, and the elastic properties need not be compatible with any of the four types of structures if only rigid body and normal modes are used.*

# III. DEVELOPMENT OF METHOD FOR SYSTEM PROCESSING

The basic system as defined by mode $[\phi]$, mass $[M]$, damping $[C]$, and stiffness $[K]$ matrices combined with a transformation matrix $[\beta]_I$ (the method of calculating $[\beta]_I$ follows) becomes a system which is attached to other systems of the composite system. A basic system may be used for more than one system.

Using the definitions:

$$[T]_1 = \left[\begin{array}{c:c} \beta_1 & 0 \end{array}\right]$$

$$[T]_2 = \left[\begin{array}{c:c} \beta_2 & 0 \end{array}\right]$$

$$\vdots$$

$$[T]_{I-1} = \left[\begin{array}{c:c} \beta_{I-1} & 0 \end{array}\right]$$

$$[T]_I = \left[\begin{array}{c:c} 0 & \diagdown 1 \diagdown \end{array}\right]$$

$$\{P\} = \begin{Bmatrix} \{I\}_{I-1} \\ \{P\}_I \end{Bmatrix}$$

the number of columns of the "0" matrices and the size of the unit matrix are equal to the number of elements in $\{P\}_I$. When $I = 2$, $[\beta]_1 = [\diagdown 1 \diagdown]$ and a recursive process will be developed to define the $[\beta]_J$ when $I > 2$.

$$\{P\}_J = [T]_J \{P\}$$

The elements of $\{P\}$ are not all independent. The dependent generalized displacements can be solved for in terms of the independent generalized displacements, making use of matching motions at connection points. There are precisely as many dependent generalized displacements as there are degrees of freedom with matching displacements. The displacements to be matched are $U_J$ and $U_I$:

$$u_J = \langle\gamma\rangle_J [\phi]_J [T]_J \{P\}$$

$$u_I = \langle\gamma\rangle_I [\phi]_I [T]_I \{P\}$$

where $\langle\gamma\rangle_J$ and $\langle\gamma\rangle_I$ are the rows of the direction cosine matrices that transform coordinates in the local coordinate system of the parts into a common system. Each $[\phi]_J$ and $[\phi]_I$ are the three rows of the modal matrix associated with the displacement being matched, and $[T]_J$ and $[T]_I$ are the transformation matrices for the systems being joined. Subtracting the coefficients of $\{P\}$, we define the row matrix $\langle\Phi\rangle$:

$$\langle\Phi\rangle = \langle\gamma\rangle_I [\phi]_I [T]_I - \langle\gamma\rangle_J [\phi]_J [T]_J$$

Repeating this process for each displacement being matched and forming a matrix of the results, we have

$$\langle\Phi\rangle = \begin{bmatrix} \langle\Phi\rangle_1 \\ \langle\Phi\rangle_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad [\phi]\{P\} = 0$$

Let $[\Phi_D]$ = those columns of $[\Phi]$ associated with redundant generalized displacements $\{D\}$. and $[\Phi_I]$ is the remaining matrix of columns of $[\Phi]$ associated with $\{I\}$. After rearrangement,

$$\left[\begin{array}{c:c} [\Phi_I] & [\Phi_D] \end{array}\right] \begin{Bmatrix} \{I\} \\ \{D\} \end{Bmatrix} = 0$$

Expanding, we have

$$[\Phi_I]\{I\} + [\Phi_D]\{D\} = 0$$

Therefore:

$$\{D\} = -[\Phi_D]^{-1}[\Phi_I]\{I\}_I$$

and

$$\begin{Bmatrix} \{I\}_I \\ \{D\} \end{Bmatrix} = [T]\{I\}_I\text{, where } [T] = \begin{bmatrix} \diagdown 1 \diagdown \\ \hdashline -[\Phi_D]^{-1}[\Phi_I] \end{bmatrix}$$

In order to invert the matrix $[\Phi_D]$, it must be square and non-singular. This condition is satisfied if the imposed matching motions are necessary to ensure the required matching motions (problems sometimes arise when attaching systems at two restrained points along the line connecting them). The number of matching displacements must match the number of dependent generalized displacements. For equilibrium of all parts of the structure, the set of generalized displacements chosen to be independent must be in fact independent on each application of this process. An example where this is not satisfied is given. Two planar systems are attached at three points along a line. Each system has only one mode with relative motion between the three points. Both of these modes cannot be independent.

The matrix $[T]$ after rearrangement into the original order of generalized coordinates is the transformation which reduces the degree of the problem from the total number of modes of the parts to the number of independent modes of the parts. The matrices $[T]_I$ are postmultiplied by this transformation, and the resulting matrices $[\beta]_I$ are stored.

The process just described is followed, as one system is added at a time to the current composite system, and the process is repeated for each system to be added starting with the second system.

After the compatibility conditions between all systems have been imposed, the equations of equilibrium for each system can be combined in the following form as can be checked by inspection.

$$\left[\begin{array}{c|c|c} M_1 & 0 & 0 \\ \hline 0 & M_2 & 0 \\ \hline 0 & 0 & \ddots \end{array}\right]\left[\begin{array}{c} \beta_1 \\ \hline \beta_2 \\ \hline \vdots \end{array}\right]\{\ddot{I}\} + \left[\begin{array}{c|c|c} C_1 & & \\ \hline & C_2 & \\ \hline & & \ddots \end{array}\right]\left[\begin{array}{c} \beta_1 \\ \hline \beta_2 \\ \hline \vdots \end{array}\right]\{\dot{I}\} + \left[\begin{array}{c|c|c} K_1 & & \\ \hline & K_2 & \\ \hline & & \ddots \end{array}\right]\left[\begin{array}{c} \beta_1 \\ \hline \beta_2 \\ \hline \vdots \end{array}\right]\{I\} = \left\{\begin{array}{c} F_1 \\ \hline F_2 \\ \hline \vdots \end{array}\right\}$$

To preserve symmetry, both sides of the equation are premultiplied by

$$\left[\begin{array}{c} \beta_1 \\ \hline \beta_2 \\ \hline \vdots \end{array}\right]^T$$

as is done for all transformations.

The equations of equilibrium for the composite can now be written in the following form:

$$[\mathcal{M}]\{\ddot{I}\} + [\mathcal{C}]\{\dot{I}\} + [\mathcal{K}]\{I\} = \{\mathcal{F}\}$$

where

$$[\mathcal{M}] = \sum_I [\beta]_I^T [M]_I [\beta]_I$$

$$[\mathcal{C}] = \sum_I [\beta]_I^T [C]_I [\beta]_I$$

$$[\mathcal{K}] = \sum_I [\beta]_I^T [K]_I [\beta]_I$$

$$\{\mathcal{F}\} = \sum_I [\beta]_I^T \{F\}_I$$

# IV. DEVELOPMENT OF METHOD FOR COMPOSITE PROCESSING

The equilibrium equations are now written in terms of the independent generalized coordinates $\{I\}$ which are partitioned into the six rigid body modes $\{R\}$ of the principal system and the remaining elastic modes $\{E\}$ with the generalized mass, damping, stiffness and load matrices similarly partitioned:

$$\left[\begin{array}{c|c} \mathcal{M}_{RR} & M_{RE} \\ \hline \mathcal{M}_{ER} & \mathcal{M}_{EE} \end{array}\right] \left\{\begin{array}{c} \ddot{R} \\ \hline \ddot{E} \end{array}\right\} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & C_{EE} \end{array}\right] \left\{\begin{array}{c} \dot{R} \\ \hline \dot{E} \end{array}\right\} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathcal{K}_{EE} \end{array}\right] \left\{\begin{array}{c} R \\ \hline E \end{array}\right\} = \left\{\begin{array}{c} Q_R \\ \hline Q_E \end{array}\right\}$$

A transformation is introduced which reduces the size of the problem by six (the number of rigid body modes of the principal system). This transformation depends on the loading type:

$$\left\{\begin{array}{c} \ddot{R} \\ \hline \ddot{E} \end{array}\right\} = [T_R]\,\{\ddot{E}\} + \left\{\begin{array}{c} R_C \\ \hline 0 \end{array}\right\}$$

For loading type No. 1 where rigid body accelerations of the principal system are given:

$$[T_R] = \left[\begin{array}{c} \diagdown\; 0 \\ \hline 1 \;\diagdown \end{array}\right] \text{ and } \{\ddot{R}_C\} = \{\ddot{R}\}$$

As can be seen by expanding the set of equations associated with $\{E\}$ and moving the term $[\mathcal{M}_{ER}]\,\{\ddot{R}\}$ to the right side of the equation, the elastic displacements can be found directly using the effective forcing function $-\,[\mathcal{M}_{ER}]\,\{\ddot{R}\}$ and the transformation is an identity.

$$[\mathcal{M}_{EE}]\,\{\ddot{E}\} + [C_{EE}]\,\{\dot{E}\} + [\mathcal{K}_{EE}]\,\{\ddot{E}\} = -\,[\mathcal{M}_{ER}]\,\{R\}$$

For loading type No. 2 where loads at a point are given for a free composite system:

$$[T_R] = \left[\begin{array}{c} -\,[\mathcal{M}_{RR}]^{-1}\,[\mathcal{M}_{RE}] \\ \hline 1 \end{array}\right] \text{ and } \{\ddot{R}_C\} = [\mathcal{M}_{RR}]^{-1}\,\{Q_R\}$$

In order to invert $[M_{RR}]$ it must be non-singular; this may require associating mass with some restrained degrees of freedom.

From the partitioned equations associated with the rigid body modes:

$$[\mathcal{M}_{RR}]\,\{\ddot{R}\} + [\mathcal{M}_{RE}]\,\{\ddot{E}\} = \{Q_R\}$$

Therefore: $\{\ddot{R}\} = [\mathcal{M}_{RR}]^{-1}\left\{\{Q_R\} - [\mathcal{M}_{RE}]\,\{E\}\right\}$

Substituting this in the set of equations associated with the elastic modes:

$$[\mathcal{M}_{ER}]\,[\mathcal{M}_{RR}]^{-1}\left\{\{Q_R\} - [\mathcal{M}_{RE}]\,\{E\}\right\} + [\mathcal{M}_{EE}]\,\{E\} + [C_{EE}]\,\{E\} + [\mathcal{K}_{EE}]\,\{E\} = \{Q_E\}$$

Rearranging terms, an equation solvable for $\{E\}$ is found:

$$\left[[\mathcal{M}_{EE}] - [\mathcal{M}_{ER}]\,[\mathcal{M}_{RR}]^{-1}\,[\mathcal{M}_{RE}]\right]\{\ddot{E}\} + [C_{EE}]\,\{\dot{E}\} + [\mathcal{K}_{EE}]\,\{E\} = \left\{\{Q_E\} - [\mathcal{M}_{ER}]\,[\mathcal{M}_{RR}]^{-1}\,\{Q_R\}\right\}$$

Making use of the appropriate transformation and the following definitions

$$[M] = [T_R]^T\,[\mathcal{M}]\,[T_R],\ [C] = [T_R]^T\,[C]\,[T_R],$$

$$[K] = [T_R]^T\,[\mathcal{K}]\,[T_R] \text{ and } \{Q\} = [T_R]^T\,\{Q\}$$

the equations of equilibrium can be written in the following form:

$$[M]\,\{\ddot{E}\} + [C]\,\{\dot{E}\} + [K]\,\{E\} = \{Q\}$$

While the matrix coefficients can be obtained using $[T_R]$ as a transformation matrix, they are developed directly in the program. The transformation can be seen to give the correct value of $\{\ddot{E}\}$ and $\{\ddot{R}\}$ by inspection.

The $\{R_C\}$ matrix represents the nonfrequency dependent portion of the generalized coordinates.

At this point, the undamped eigenvalue problem is solved and the eigenvectors associated with the lowest frequency modes are used to form a transformation matrix $[V_U]$ which reduces the order of the problems to a level, such that the damped eigenvalue problem can be solved with an existing program. Since the mass matrix is non-diagonal, a triangular matrix $[F]$ is found, such that $[F]^T\,[F] = [M]$ and the inverse of this matrix (also triangular) is used as a transformation matrix which transforms the mass matrix into a unit matrix prior to solving the undamped eigenvalue problem. A simple direct method of calculating both $[F]$ and $[F]^{-1}$ is given in "Theory of Mechanical Vibration" by Kin N. Tong.

Since $\{E\} = [F^{-1}]\,[V_U]\,\{E_U\}$ and using the following definitions

$$\lceil M_U \rfloor = [V_U]^T\,[F^{-1}]^T\,[M]\,[F^{-1}]\,[V_U]$$

$$[\,C_U\,] = [V_U]^T\,[F^{-1}]^T\,[C]\,[F^{-1}]\,[V_U]$$

$$\lceil K_U \rfloor = [V_U]^T\,[F^{-1}]^T\,[K]\,[E^{-1}]\,[V_U]$$

$$\{\,Q_U\,\} = [V_U]^T\,[F^{-1}]^T\,\{Q\}$$

we observe that $[M_U] = \lceil 1 \rfloor$ and $\lceil K_U \rfloor = \lceil \omega^2 \rfloor$.

The equations of equilibrium can be written in the following form:

$$\lceil 1 \rfloor\,\{\ddot{E}_U\} + [C_U]\,\{\dot{E}_U\} + \lceil \omega^2 \rfloor\,\{E_U\} = \{Q_U\}$$

The damped eigenvectors $[V_D]$ are used to transform the equations of equilibrium into a form in which the variables are separated.

Since $\{E_U\} = [V_D]\,\{z\}$ and using the following definitions

$$\lceil R \rfloor = 2\,\lceil \alpha \rfloor\,[V_D]^T\,[V_D] + [V_D]^T\,[C_U]\,[V_D]$$

$$\{D\} = [V_D]^T\,\{Q_U\}$$

$$\lceil R \rfloor\,\{\dot{Z}\} - \lceil R \rfloor\,\lceil \alpha \rfloor\,\{Z\} = \{D\}$$

This last procedure is described by K. A. Foss in "Coordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," *Journal of Applied Mechanics*, Vol. 25, 1958. $\lceil \alpha \rfloor$ is the matrix of complex eigenvalues and $[V_D]$ is the associated eigenvector matrix. $[C_U]$ and $\{Q_U\}$ are the damping and loading matrices after transformation into the undamped normal mode coordinates $\{E_U\}$.

## V. DEVELOPMENT OF RESPONSE CALCULATION

With the variables separated, the loads given, and using the equality $Z = j\omega Z$, the generalized coordinates can be found from the equation:

$$Z_n = \frac{D_n}{R_{nn}}\,\frac{1}{j\omega - \alpha n}$$

The matrices $\lceil R \rfloor$, $\lceil \alpha \rfloor$, $[\phi]$, $[\beta]_I$, $[T]$, $[M_{ER}]$

where $[T] = [T_R]\,[F^{-1}]\,[V_U]\,[V_D]$

have been saved on tape from previous computations.

If loading consists of an arbitrary sinusoidal acceleration of the rigid base of the primary system (Type 1 loading),

$$\{Q\} = -\,[M_{ER}]\,\{R\}$$

If loading consists of a sinusoidal force at a point on an unrestrained structure (Type 2 loading),

$$\{Q\} = [\beta]^T\,[\phi]^T\,\{f\}$$

It is observed that:

$$\{D\} = [T]\,\{Q\}$$

These participation factors can be transformed into the displacement of points within a system as shown in the following equation:

$$\{\ddot{u}\} = [\phi]\,[\beta] \left\{ [T]\,\{\ddot{Z}\} + \left\{ \frac{\{\ddot{R}_C\}}{0} \right\} \right\} = [\phi]\,[\beta]\,\{\,\ddot{I}\,\}$$

The accelerations of up to 10 control degrees of freedom are calculated as a function of frequency and the modulus of these accelerations is plotted. The composite-system generalized accelerations $\{\ddot{I}\}$ corresponding to the frequency which maximized control accelerations are punched by the computer on IBM cards.

# VI. DEVELOPMENT OF POINT ACCELERATION AND MEMBER STRESS CALCULATION

The acceleration of each degree of freedom in a system is calculated using the composite system generalized accelerations $\{\ddot{I}\}$ which were punched on cards in the response calculations.

Inertial loads are found using these accelerations. These loads are applied to the basic system and the resulting deflections are superimposed on the constraint mode deflections, rigid body mode deflections and deflections due to loads at attachment points which are found as follows.

Defining the matrices $[R_0]$ and $\{R_A\}$ as the matrix of loads associated with attachment modes and a matrix of unknown multipliers, the loads at attachment points $([R_0]\ \{R_A\})$ are found from the equations of equilibrium associated with the attachment modes which are

$$\left[ M_{AR} \mid M_{AC} \mid M_{AA} \mid M_{AN} \right] \begin{Bmatrix} \ddot{P}_R \\ \hline \ddot{P}_C \\ \hline \ddot{P}_A \\ \hline \ddot{P}_N \end{Bmatrix} + \lceil C_{AA} \rfloor \{\dot{P}_A\}$$

$$+ \left[ K_{AA} \mid K_{AN} \right] \begin{Bmatrix} P_A \\ \hline P_N \end{Bmatrix} = \{F_A\}$$

$$\text{where } \{F_A\} = [\phi_A]^T [R_0] \{R_A\} + [\phi_A]^T \{f_L\}$$

$$= [K_{AA}] \{R_A\} + [\phi_A]^T \{f_L\}$$

$$\text{since } [R_0] = [k] [\phi_A],\ [K_{AA}] = [\phi_A]^T [k] [\phi_A]$$

and $\{f_L\}$ is the forcing load matrix for Type 2 loading. Notice that $[K_{AN}]$ is not equal to $[0]$ since normal modes can have displacements at attachment mode loading points. The deflection due to the loads at attachment points is $[\phi_A]\ \{R_A\}$ since $[k]\ [\phi_A] = [R_0]$.

These deflections are multiplied by $-\omega^2$ for comparison with the modal accelerations.

Deflections associated with either set of accelerations are used to calculate member loads. The $12 \times 12$ member stiffness matrix $[K]$ is developed as was done in JPL TM 33-75.

$$\{R\} = [K]\ \{u\}$$

where $\{u\}$ is the set of deflection of both ends of the member, and $\{R\}$ is the corresponding set of reactions. The member loads are dot and cross products of the load vector at the end with the direction cosine vector $\overline{\gamma}$ of the member.

| | | | | |
|---|---|---|---|---|
| Axial load | $(P)$ | = | $\overline{\gamma} \cdot \overline{R}$ | (first 3 elements of $R$) |
| Torsional load | $(T)$ | = | $\overline{\gamma} \cdot \overline{R}$ | (second 3 elements of $R$) |
| Shear load | $(V)$ | = | $\lvert \overline{\gamma} \times \overline{R} \rvert$ | (first 3 elements of $R$) |
| Moment at first end | $(M_A)$ | = | $\lvert \overline{\gamma} \times \overline{R} \rvert$ | (second 3 elements of $R$) |
| Moment at other end | $(M_B)$ | = | $\lvert \overline{\gamma} \times \overline{R} \rvert$ | (fourth 3 elements of $R$) |

The rigid body deflections are included only to allow comparison between the point accelerations which are based on modal accelerations and those based on inertial loading; they do not affect member loads. The inertial loading method of determining member loads implies no member loads due to internal damping. An alternate method of determining member loads directly from modal deflection is also provided for highly damped structures or structures with point loads at points without attachment mode loads which are not properly loaded by the mass acceleration method.

# VII. PROGRAMMING

## A. Input Format

### Basic System Input

**Control Card 1**

$$\frac{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5}{(5I8)}$$

$N_1$ = structure type

= 1 for pin-jointed structure

= 2 for rigid-jointed structure

= 3 for planar structure loaded in plane

= 4 for planar structure loaded out of plane

$N_2$ = record number at which basic system information is to be loaded on logical Tape 11

$N_3$ = number of links in rigid body

$N_4$ = number of normal modes to be used

$N_5$ = 1 if viscous damping is present

= 0 otherwise

**Control Card 2**

$$\text{Name}\ \frac{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7}{(2X, A6, 7I8)}$$

Name = alphanumeric run number (6 characters or less)

$N_1$ = number of joints in basic system

$N_2$ = number of members in basic system (may be zero)

$N_3$ = number of attachment modes (may be zero)

$N_4$ = 1 if weight cards included

= 0 otherwise

$N_5$ = number of joints with one or more restrained degrees of freedom

$N_6$ = number of degrees of freedom per joint

$N_7$ = 1 if stiffness matrix output desired

= 0 otherwise

**Control Card 3**

$$\frac{N_1 \quad N_2 \quad E \quad \nu}{(2I8, 2F8.2)}$$

$N_1$ = number of stiffness matrix elements to be altered

$N_2$ = 1, if rigid body modes to be calculated

= 0 otherwise

$E$ = modulus of elasticity (in thousands of pounds/sq in.)

$\nu$ = Poisson's ratio

**Joint Cards**

One card is for each joint, with cards in monotonic increasing order.

$$\frac{JT \quad LK \quad X_1 \quad X_2 \quad X_3}{(2I8, 3F8.2)}$$

$JT$ = joint number

$LK$ = link containing $JT$

($LK$ = 0 for points on main body)

$(X_1, X_2, X_3)$ = spatial coordinates of joint

**Member Property Cards**

One card must be supplied for each member.

$$\frac{JTA \quad JTB \quad A_1 \quad A_2 \quad A_3}{(2I8, 3F8.2)}$$

$JTA$ = joint number of end one of member

$JTB$ = joint number of other end of member

$A_1$ = member area if $A_3 \neq 0$

= outside diameter if $A_3 = 0$

$A_2$ = bar moment of inertia if $A_3 \neq 0$

= wall thickness if $A_3 = 0$

$A_3$ = bar $K$ or zero if area, moment of inertia and torsional stiffness are to be computed from outside diameter and wall thickness. Only a positive number need be supplied if area given and structure of type 1 or 3.

**Restraint Cards**

| $JT$ $N_1$ $N_2$ $\cdots$ $N_k$ |
|---|
| $(7I8)$ |

$N_i$ = 1, if restrained against motion in subscripted degree of freedom (see weight input for order)

= 0 if unrestrained

= 2 if constraint mode degree of freedom

$k$ = number of degrees of freedom per joint

**Stiffness Matrix Elements**

| $i$ $j$ $\Delta K_{ij}$ |
|---|
| $(2I8, F8.2)$ |

$(i\ \ j)$ = row and column, respectively, of element in uncontracted stiffness matrix (inserted before rows and columns have been deleted to account for restraints)

$\Delta K_{ij}$ = incremental change to element $K_{ij}$ (in lb/in., in.-lb/rad or lb) of original stiffness matrix. The new element $K_{ij} = K_{ij} + \Delta K_{ij}$

One of these cards must be supplied for each stiffness matrix element to be altered. This information is to be supplied only if $N_1$ on control card 3 is greater than 0.

**Weight Cards**

One card must be supplied for each joint.

| $JT$ BLANK $W_1$ $W_2$ $W_3$ $W_4$ $W_5$ $W_6$ |
|---|
| $(2I8, 6F8.2)$ |

This input will vary depending on the structure type (units are lb and in.$^2$ lb).

$JT$ = joint number

**Type 1 and Type 2 Structures**

$W_1$ = weight in $X_1$ direction

$W_2$ = weight in $X_2$ direction

$W_3$ = weight in $X_3$ direction

$W_4$ = moment of inertia about $X_1$ axis

$W_5$ = moment of inertia about $X_2$ axis

$W_6$ = moment of inertia about $X_3$ axis

**Type 3 Structures**

$W_1$ = weight in $X_1$ direction

$W_2$ = weight in $X_2$ direction

$W_3$ = moment of inertia about $X_3$ axis

$W_4$ = weight in $X_3$ direction

$W_5$ = moment of inertia about $X_1$ axis

$W_6$ = moment of inertia about $X_2$ axis

**Type 4 Structure**

$W_1$ = weight in $X_3$ direction

$W_2$ = moment of inertia about $X_1$ axis

$W_3$ = moment of inertia about $X_2$ axis

$W_4$ = weight in $X_1$ direction

$W_5$ = weight in $X_2$ direction

$W_6$ = moment of inertia about $X_3$ axis

**Attachment Mode Cards**

One card is supplied for each load.

| $JT$ $N$ $F_1$ $F_2$ $\cdots$ $F_k$ |
|---|
| $(2I8, 6F8.2)$ |

$JT$ = joint where load is applied

$N$ = dummy No. not to be used

$F_i$ = load (lb or in.-lb) in $i^{\text{th}}$ degree of freedom (see weight input for order)

$k$ = number of degrees of freedom per joint

**Normal Mode Cards**

One card per unrestrained degree of freedom. If 1 in second field of first card, these cards are output from "STIF-EIG" (see JPL TM No. 33-75, as altered) and are added without change except for elimination of cards corresponding to restraints.

| $U_1$ $U_2$ $U_3$ $U_4$ $U_5$ $U_6$ |
|---|
| $(6E12.5)$ |

$U_i$ = motion (in. or rad) of degree of freedom in $i^{\text{th}}$ mode.

**Rigid Body Information**

This information is supplied in sets of three cards, one set per link, and are in serial order according to link.

**Card 1**

$\dfrac{J \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6}{(7I8)}$

**Card 2**

$\dfrac{K \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6}{(7I8)}$

**Card 3**

$\dfrac{L \quad I_1 \quad I_2 \quad I_3 \quad I_4 \quad I_5 \quad I_6}{(7I8)}$

The total number of "1's" on each set of 3 cards (exclusive of joint numbers) should = 6.

$J, K$ = joints defining hinge. If $K = 0$, the program will assume one joint defined link motion.

$C_m$ = 1, if there is common motion between the link and the main body at the joint in the $m^{\text{th}}$ direction.

= 0 otherwise

$I_n$ = 1, if the link has an independent degree of freedom in the $n^{\text{th}}$ direction.

= 0 otherwise

$L$ = dummy number not to be used

**Modifications to Generalized Mass Matrix**

$\dfrac{J, K, \Delta M}{(2I8, E16.4)}$

$J$ = row of element in matrix

$K$ = column of element in matrix

$\Delta M$ = addition to element ($M_{JK} = M_{JK} + \Delta M$).

**Frequencies**

$\dfrac{F_1 \quad F_2 \quad \cdots \quad F_6}{(9F8.2)}$

$F_i$ = frequency of $i^{\text{th}}$ normal mode (cycles per second)

**Modifications to Generalized Stiffness Matrix**

$\dfrac{M \quad N \quad \Delta K}{(2I8, E16.4)}$

$M$ = row of element in matrix

$N$ = column of element in matrix

$\Delta K$ = addition to element ($K_{MN} = K_{MN} + \Delta K$).

**Viscous Damping Coefficients (if $N_5$ on card 1 = 1)**

$\dfrac{C_{11} \quad C_{22} \quad \cdots}{(9F8.2)}$ (1 number per mode, 9 per card)

$C_{KK}$ = $K^{\text{th}}$ diagonal element of damping matrix (in.-lb/sec) order of elements rigid body mode, constraint mode, attachment mode and normal modes.

## System Input

**Control Card 1 (first card if no basic system processing this run)**

$\dfrac{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5}{(5I8)}$

$N_1 = 1$

$N_2 = N_3 = N_4 = N_5 = 0$

**Control Card 4**

$\dfrac{N_1 \quad N_2 \quad N_3}{(3I8)}$

$N_1$ = number of basic systems to be used to define systems.

$N_2$ = number of systems to be used in total structure.

$N_3$ = number of degrees of freedom to be deleted from total structure.

**Deleted Modes ($N_3$ numbers, 9 per card)**

$\dfrac{D_1 \quad D_2 \quad \cdots \quad D_k \quad \cdots}{(9I8)}$

$D_k$ = the $k^{\text{th}}$ mode to be deleted from the total structure. (Modes are eliminated as each system is constrained. The modes to be eliminated as any system is constrained must be in monotonic increasing order.)

$k$ = mode in system under consideration plus total of all preceding systems (order of modes in a system is rigid body, constraint, attachment, and normal).

**Basic System Location**

$\dfrac{L_1 \quad L_2 \quad \cdots \quad L_k \quad \cdots}{(9I8)}$ ($N_1$ numbers, 9 per card must be in monotonic increasing order)

$L_k$ = record number on Tape $A6$ of $k^{\text{th}}$ basic system

**Basic System Number of System**

$$\frac{M_1 \quad M_2 \quad \cdots \quad M_k \quad \cdots}{(9I8)} \quad (N_2 \text{ numbers, 9 per card})$$

$M_k$ = basic system number of the $k^{th}$ system (First system is principal system, and more than one system may reference the same basic system.)

**Note 1**

The following information through the compatibility cards is to be supplied in sets. One set for each system ($I$) except for the first. System $I$ is the system being added to the composite.

**Control Card 5**

$$\frac{NJI,\ N\ \text{OUT}\ 1,\ N\ \text{OUT}\ 2}{(3I8)}$$

$NJI$ = number of other systems $J$ to which the $I^{th}$ system is attached. Systems $J$ are already part of the composite and system $I$ is being added.

$N$ OUT 1 $\neq 0$ causes system transformations $\beta_I$ to be printed at each step.

$N$ OUT 2 $\neq 0$ causes $\phi_D$ & $\phi_I$ to be output for system $I$.

**Note 2**

The following information through the compatibility cards is supplied in sets, one set for each $J^{th}$ system (already part of composite) to which system $I$ is attached.

**Joint Cards**

$$\frac{NJ \quad L \quad JT1 \quad JT2 \quad \cdots \quad JT_K \quad \cdots \quad JT_L}{(9I8)}$$

$NJ$ = system number of the $J^{th}$ attached system.

$L$ = number of sets of geometrical transformations of $J^{th}$ and $I^{th}$ systems.

$JT_K$ = the number of joints in the $K^{th}$ transformation of systems $J$ and $I$.

**Note 3**

The following information through the compatibility cards is to be supplied in sets, one set for each transformation of the $J$ and $I$ systems.

**Spatial Transformations (2 cards)**

$$\frac{[A(M,N),\ N = 1,2,3],\ M = 1,2,3}{(9F8.2)}$$

$A$ is transformation matrix from coordinates of $J$ or $I$ system ($\{u'\} = [A]\{u\}$, where $u$ is along system coordinates and $\{u'\}$ is along common coordinates) to a common coordinate system. Order is $A_{11}$, $A_{12}$, $A_{13}$, $A_{21}$, $A_{22}$, $A_{23}$, $A_{31}$, $A_{32}$, $A_{33}$. ($[A]\,[A]^T = \ulcorner 1 \lrcorner$.) The transformation for the $J^{th}$ system (already part of composite) is followed by that for the $I^{th}$ system (system being added).

**Compatibility Card**

$$\frac{JTJ \quad JT1 \quad N_1 \quad N_2 \quad \cdots \quad N_6}{(9I8)}$$

One of these cards must be supplied for every joint in the current transformation.

$JTJ$ = joint number in the $J^{th}$ system

$JTI$ = joint number in the $I^{th}$ system

$N_K$ = 1 if common motion between $JTI$ and $JTJ$ in the $K^{th}$ direction in the common coordinate system (each 1 corresponds to a mode in the deleted mode list).

= 0 otherwise

## Composite Input

**Control Card 1**

$$\frac{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5}{(5I8)} \quad \text{(first card if no system processing this run)}$$

$N_1$ = 2 if no system processing this run

= 0 otherwise

$N_2 = 0$

$N_3$ = 1 for type 1 loading (base acceleration of constrained composite system)

= 2 for type 2 loading (load at point of unrestrained composite system)

$N_4 = N_5 = 0$

**Control Card 7**

$$\frac{N_1 \quad N_2}{(2I8)}$$

$N_1$ = number of eigenvectors to be retained

$N_2$ = > 0 if printed displacements for all systems required

= < 0 if printed and punched displacements required

## Dynamic Response Input

**Control Card 1**

$\underset{(5I8)}{\underline{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5}}$ (first card if no composite processing this run)

$N_1 = 3$

$N_2 = N_3 = N_4 = N_5 = 0$

**Control Card 8**

$\underset{(3I8)}{\underline{N_1 \quad N_2 \quad N_3}}$

$N_1$ = number of critical points $\leq 10$

$N_2$ = 1 if base accelerations to be given

= 2 if load at points on an unrestrained composite structure to be given

$N_3$ = number of loads $\leq 10$

**System Numbers of Critical Points**

$\underset{(9I8)}{\underline{NS_1 \quad NS_2 \quad NS_3 \quad \cdots}}$

$NS_K$ = system number corresponding to $K^{th}$ critical point.

**Degree of Freedom of Critical Points**

$\underset{(9I8)}{\underline{NF_1 \quad NF_2 \quad NF_3 \quad \cdots}}$

$NF_K$ = degree of freedom of $K^{th}$ critical point (degree of freedom in basic system with 6 degrees of freedom per joint)

**Frequency Information (radians per second)**

$\underset{(3F8.2)}{\underline{\omega_0 \quad \omega_f \quad \Delta\omega}}$

$\omega_0$ = initial value of $\omega$

$\omega_f$ = final value of $\omega$

$\Delta\omega$ = step size (a value of one fifth the smallest real part of the complex eigenvalues should be satisfactory)

**Load Information**

$\underset{(I1, I7, I8, 6F8.2I8)}{\underline{I \quad S \quad F \quad L_1 \quad L_2 \quad \cdots \quad L_6 \quad IND}}$ (1 card for each load)

$I$ = 0 if not last joint-loaded; blank if last or only joint-loaded

S = system number of load point for type 2 loading

$F$ = joint number of load point for type 2 loading

$L_i$ = amplitude of sinusoidal forcing function in $i^{th}$ direction

IND = 2 if plots of response required; 0 if no plots required

## Accelerations and Member Loads Input

**Control Card 1**

$\underset{(5I8)}{\underline{N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5}}$ (first card if no dynamic response required)

$N_1 = 4$

$N_2 = N_3 = N_4 = N_5 = 0$

**Control Card 9**

$\underset{(9I8)}{\underline{N1 \quad N2 \quad NC(I), \quad I = 1, \quad N1}}$

$N_1$ = number of systems in composite

$N_2$ = number of loads

$NC(I)$ = 1, accelerations and member loads required for System $I$

= 0, not required

**Load Information**

The following information through participation factors is required for each load for which accelerations and member loads are being computed:

**Control Card 10**

$\underset{(I8)}{\underline{N \text{ TYPE}}}$

$N$ TYPE = 1 if base accelerations given

= 2 if load at points of an unrestrained composite structure given

*Joint loads* (if $N$ TYPE = 2 only) Data identical to load information

$$\frac{I \quad S \quad F \quad L_1 \quad L_2 \quad \cdots \quad L_6}{(I1, I7, I8, 6F8.0)}$$

$I$ = 0, if not last joint-loaded

= blank if last or only joint-loaded

$S$ = system number of load point

$F$ = joint number of load point

$L_i$ = amplitude of sinusoidal forcing function in $i^{th}$ direction

**Participation Factors**

$$\frac{\omega, [PR(I), I = 1, N]}{(6E12.4)}$$

$$\frac{\omega, [PI(I), I = 1, N]}{(6E12.4)}$$

$\omega$ = forcing frequency

$PR(I)$ = real part of $I^{th}$ participation factor

$PI(I)$ = imaginary part of $I^{th}$ participation factor

$N$ = number of generalized displacements in composite system

*Note 4*

These cards are punched by computer during response calculations and desired sets are hand selected.

*Note 5*

The following information to the end of this section is required for each system indicated with "1's" on control card 9.

**Number of Loadings**

$$\frac{NF}{(I8)}$$

$NF$ = number of loading conditions for System $I$

The following card is required for each load:

**Option Card**

$$\frac{N1, N2, N3}{(3I8)}$$

$N1$ = 0 if modal displacements are to be used to find member loads.

= 1 if D'Alembert loads are to be used to find member loads.

$N2$ = 0 if only member load amplitudes required.

= 1 if input member properties, direction cosines and components of member loads required also.

$N3$ = Load number (1 to $N_2$ Card 9) to be used.

### B. Output Format

1. *Basic System Output*
   a. Basic system input, except normal-modes frequencies and damping, printed
   b. Stiffness matrix printed if $N_7$ = 1 on control card 2
   c. Modal matrix $[\phi]$
   d. Generalized mass matrix (lb-sec²/in.)
   e. Generalized stiffness matrix
   f. Generalized damping

2. *System Output*
   a. System input printed
   b. $\beta_I$ of incomplete composite, $\phi_D$ and $\phi_I$ printed on demand (control card 5)
   c. Transformation matrix $[\beta]_I$ for each system printed

3. *Composite System Output*
   a. Composite system input printed
   b. Generalized mass matrix $[m]$
   c. Generalized stiffness matrix $[k]$
   d. Generalized damping matrix $[c]$
   e. Undamped eigenvalues (equal to generalized stiffness matrix diagonal elements)
   f. Undamped modal matrix $[T_R F^{-1} V_U]$ printed
   g. Generalized damping matrix in terms of undamped normal modes
   h. Damped eigenvalues $\lceil \alpha \rfloor$ and associated eigenvectors (complex) $[V_D]$

i. Diagonal elements of combined mass and damping matrix $\lceil R \rfloor$ and maximum normalized off-diagonal absolute value

j. Damped modal matrix $[T_R F^{-1} V_U V_D]$ real elements of each vector followed by imaginary elements

4. *Load Dependent Output** (repeated for each loading)

a. Dynamic loading input is printed

b. Load vector $\{D\}$ (complex)

c. Acceleration of control point at each increment of frequency ($\ddot{U}$ complex); magnitude also printed and the magnitude plotted if IND = 2 on load card

d. Participation factors for frequency which maximized absolute values of control point acceleration (complex values of composite system accelerations $\{\ddot{P}\}$ are also output on IBM cards); (c.) and (d.) are repeated for each control point

5. *Accelerations and Member Loads*

a. Load factors $\{R_A\}$

b. Point accelerations $\{\ddot{U}\}$ and $\{\ddot{U}'\}$

c. Input member properties, direction cosines, and components of member loads (if $N3 = 1$ on load card)

d. Member load amplitudes

## C. Limitations

1. *Basic Systems*

a. Degree of freedom of structure after deletion of restrained degrees of freedom $\leq 130$; also $\geq 1$

b. Joints in structure $\leq 60$

c. Members in structure $\leq 200$

*Only amplitude of $D$, $\ddot{U}$ and $\ddot{P}$ is given; the term $e^{j\omega t}$ is understood.

d. Components of restraint $\leq 200$

e. Total number of modes $\leq 72$

f. Number of constraint modes $\leq 20$

g. Attachment modes $\leq 60$

h. Rigid body modes $\leq 26$

i. Structures, having links hinged to each other, prohibited

j. No way to combine symmetric and antisymmetric modes of a symmetric structure if either constraint modes or attachment modes are required, as the constraints are different.

k. Modal damping must be a real diagonal matrix.

2. *System*

a. The total number of compatibility conditions between composite system and system $\leq 45$ (one mode eliminated for each compatibility condition).

b. The principal system can have only 6 rigid body modes.

3. *Composite System*

a. Total number of modal vectors $\leq 100$ at any time (number = total in all systems already added, less number of restraints already imposed)

b. Total number of systems $\leq 30$

c. Total number of basic systems $\leq 30$

d. Specified dependent modes must be such that the remaining set is, in fact, a set of independent modes

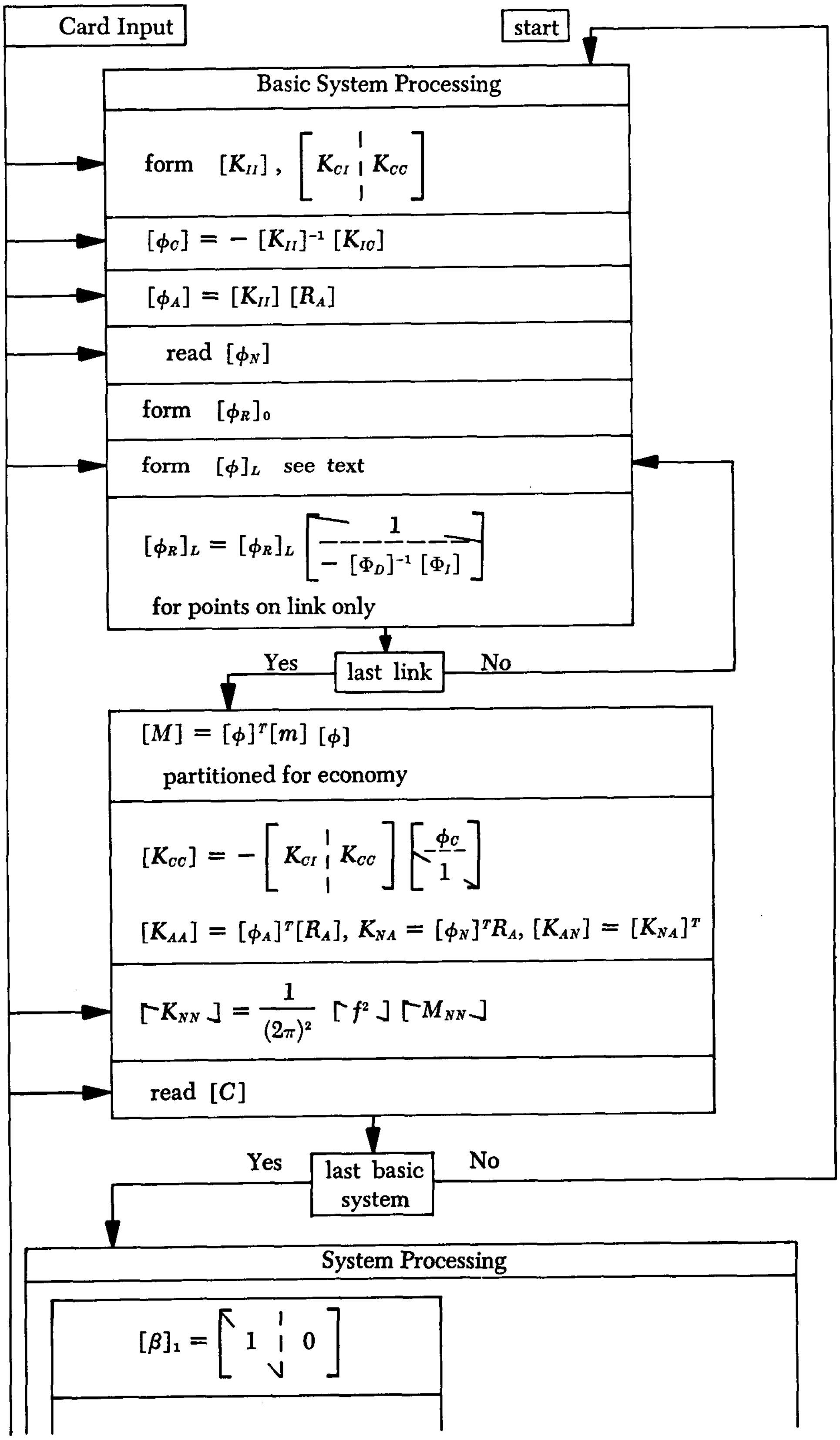
e. Total number of restraints $\leq 250$

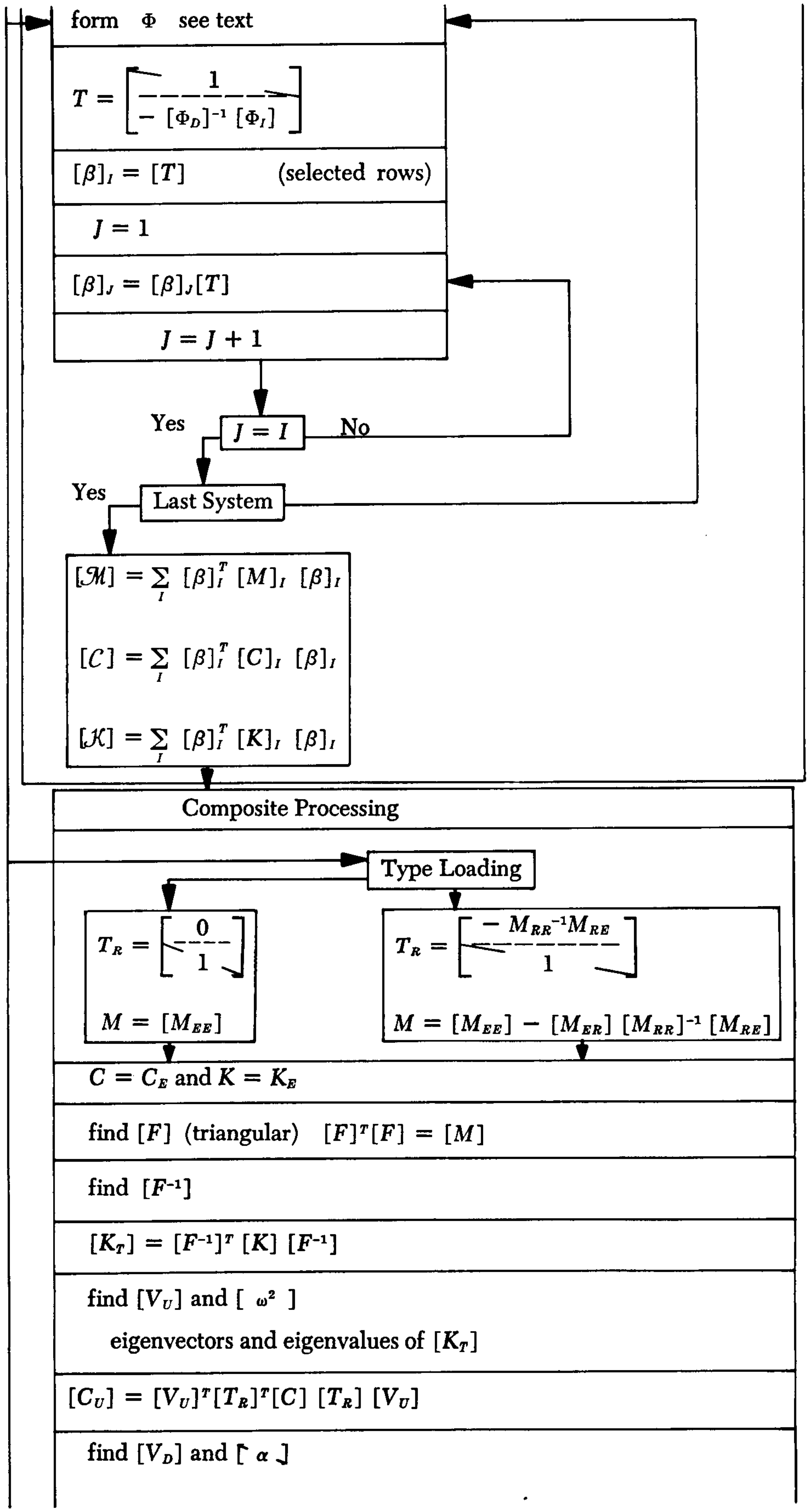
f. Rigid body modes of principal system cannot be used as dependent modes and rigid body modes of other systems must be used as dependent modes

g. Number of undamped modes retained $\leq 50$

### D. Flow Chart

*Note:* Reordering of rows and columns and deletion or expansion of restrained degrees of freedom are not shown.




form $\Phi$ see text
$T = \left[\begin{array}{c} 1 \\ \hline -[\Phi_D]^{-1}[\Phi_I] \end{array}\right]$
$[\beta]_I = [T]$ (selected rows)
$J = 1$
$[\beta]_J = [\beta]_J[T]$
$J = J + 1$
Yes
$J = I$
No
Yes
Last System
$[\mathcal{M}] = \sum_I [\beta]_I^T [M]_I [\beta]_I$
$[\mathcal{C}] = \sum_I [\beta]_I^T [C]_I [\beta]_I$
$[\mathcal{K}] = \sum_I [\beta]_I^T [K]_I [\beta]_I$
Composite Processing
Type Loading
$T_R = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$
$M = [M_{EE}]$
$T_R = \left[\begin{array}{c} -M_{RR}^{-1}M_{RE} \\ \hline 1 \end{array}\right]$
$M = [M_{EE}] - [M_{ER}][M_{RR}]^{-1}[M_{RE}]$
$C = C_E$ and $K = K_E$
find $[F]$ (triangular) $[F]^T[F] = [M]$
find $[F^{-1}]$
$[K_T] = [F^{-1}]^T [K] [F^{-1}]$
find $[V_U]$ and $[\ \omega^2\ ]$
eigenvectors and eigenvalues of $[K_T]$
$[C_U] = [V_U]^T[T_R]^T[C][T_R][V_U]$
find $[V_D]$ and $\lceil \alpha \rfloor$

damped eigenvectors and eigenvalues

$\lceil R \rfloor = 2 \lceil \alpha \rfloor [V_D]^T[V_D] + [V_D]^T[C_U][V_D]$

$[T] = [T_R][F^{-1}][V_U][V_D]$

Response Calculation

Type Loading

1

read $\{\ddot{R}\}$

$\{Q\} = -[M_{ER}]\{\ddot{R}\}$

$\{\ddot{R}_C\} = \{\ddot{R}\}$

2

read $\{f\}$

$\{Q\} = [\beta]^T[\phi]^T\{f\}$

$\{\ddot{R}_C\} = [M_{RR}]^{-1}\{Q_R\}$

$\{D\} = [T]\{Q\}$

$$\ddot{Z}_n = -\omega^2\left(\frac{D_n}{R_{nn}}\,\frac{1}{j\omega - \alpha n}\right)$$

$$\{\ddot{u}\} = [\phi]\,[\beta]\left\{[T_R]\,\{\ddot{z}\} + \begin{Bmatrix}\ddot{R}_C\\ \hline 0\end{Bmatrix}\right\}$$ (plot)

selected rows of $\phi$ & associated $\beta$

punch $\{\ddot{P}\} = [T_R]\,\{\ddot{z}\} + \begin{Bmatrix}R\\ \hline 0\end{Bmatrix}$ associated with peak values

Point accelerations and member loads

$\{\ddot{u}\} = [\phi]\,[\beta]\,\{\ddot{P}\}$

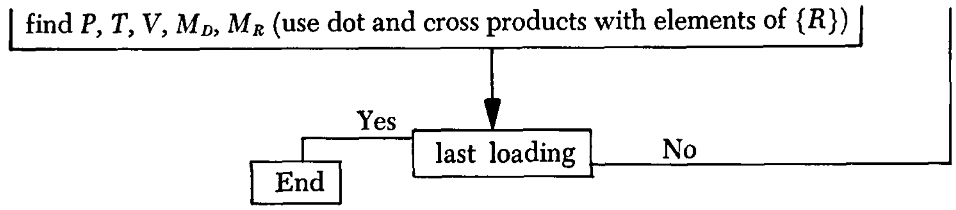
$\{f_m\} = \lceil m \rfloor\,\{\ddot{u}\}$

$$\{R_A\} = [K_{AA}]^{-1}\left\{[M_{AX}]\,\{\ddot{P}\} - \frac{j}{\omega}\lceil C_{AA} \rfloor\,\{\ddot{P}\} - \frac{1}{\omega^2}[K_{AX}]\,\{\ddot{P}\} - [\phi_A]^T\,\{f_L\}\right\}$$

$\{f\} = \{f_m\} + [R_0]\,\{R_A\}$

$\ddot{u}' = -\omega^2\,[K]^{-1}\,\{f\}$

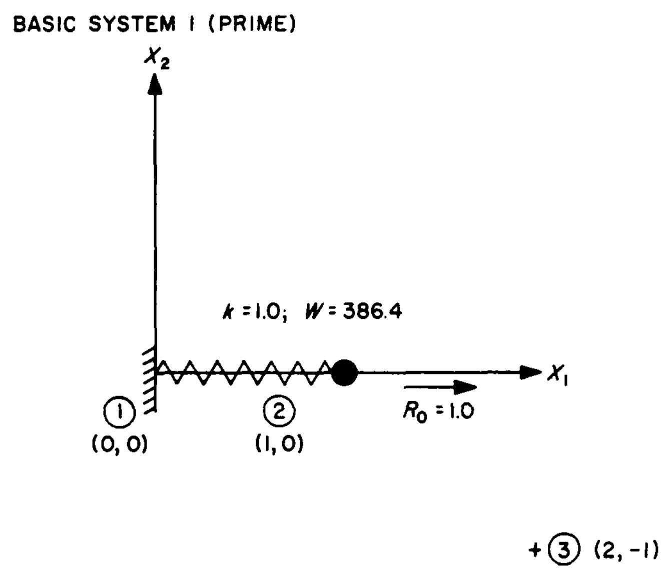
$\{R\} = -\frac{1}{\omega^2}[K]\,\{\ddot{u}\}$ for each member



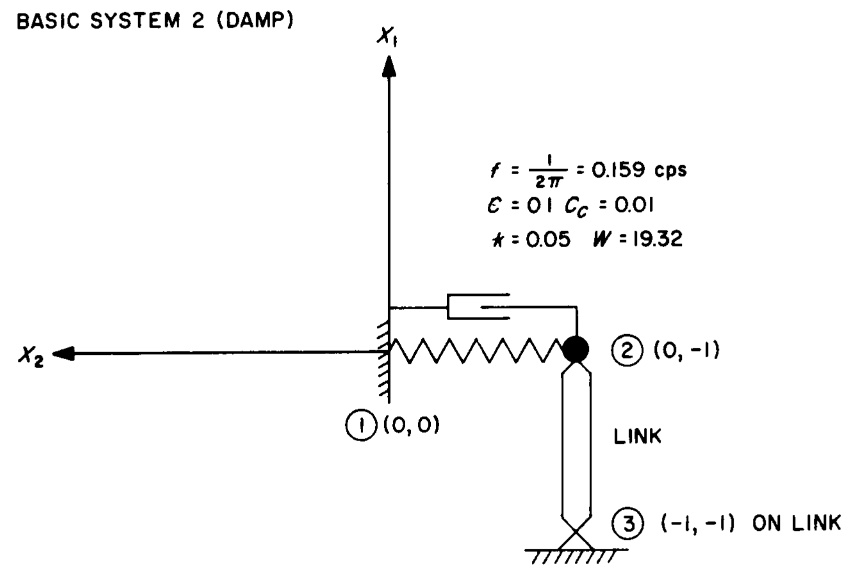

# VIII. EXAMPLE PROBLEM

An example problem is given to demonstrate the program. The problem chosen is the damped vibration absorber with two degrees of freedom. There is great flexibility available in the solution to this problem, and the arrangement shown is not chosen for its simplicity, but to illustrate some of the novel aspects of the program. The link was used as a part of the second system only to demonstrate links. In practice, it would have been simpler to merely restrain the second point in the $X_3$ direction and omit the third point. The coordinate system at the junction of the two systems was chosen only to show that it was not necessary to use the coordinate system of either part. This property is useful in sliding joints of arbitrary direction. A normal mode was used in addition to the rigid body modes for the second system. A constraint mode or an attachment mode would have done equally as well. Since as many independent modes were used as there are total degrees of freedom in the system, the results should be exact and do in fact agree with the results of another method of solution for this simple problem given in "Mechanical Vibration" by Den Hartog (page 93, 4th Ed.). Points based on the equation given are plotted on the output curve.

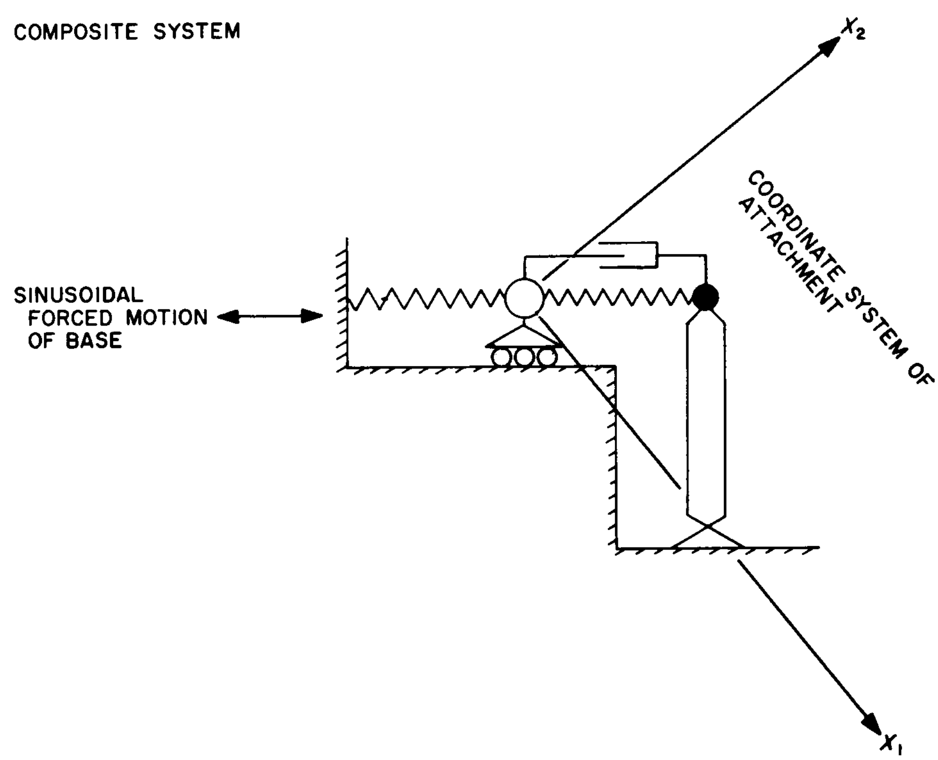
Note that both basic systems and the composite systems (see Sketch No. 2, 3, and 4) were processed and the dynamic response calculated in the same run (1 in first field, and 0 in second field of the first control card following the second system). The program was run a second time for the point accelerations and member stresses. This procedure was followed because card output of the first run was part of the input for the second run. The option of using modal accelerations instead of accelerations calculated by the mass acceleration method to calculate member loads was used, since the second system was highly damped. The comparison of accelerations as calculated by each method is good for the undamped first system and bad (as expected) for the highly damped second system.




**Sketch No. 2**




**Sketch No. 3**




**Sketch No. 4**

### A. *Input Data*

The following pages are facsimiles of printout input data.

BASIC SYSTEM INPUT:

```
    2      1      0      0      0                          CONTROL CARD NO. 1
PRIME      3      0      1      1      3      6      0     CONTROL CARD NO. 2
    1      1      0      0                                 CONTROL CARD NO. 3
    1        0.     0.      0.  ]
    2        1.     0.      0.  ]                          POINT COORDINATE CARDS
    3        2.    -1.      0.  ]
    1      1      1      1      1      1      1  ]
    2      0      1      1      1      1      1  ]         CONSTRAINT CARDS
    3      1      1      1      1      1      1  ]
    7      7     1.0                                       MODIFICATION TO STIFFNESS MATRIX
    1        0.     0.      0.      0.      0.      0. ]
    2        386.4  386.4   386.4   0.      0.      0. ]   WEIGHT CARDS
    3        0.     0.      0.      0.      0.      0. ]
    2            1.0      0.0      0.0                     ATTACHMENT MODE CARD

    1      2      1      1      1                          CONTROL CARD NO. 1
 DAMP      3      1      0      1      3      3      1     CONTROL CARD NO. 2
    1      1  1000.     0.                                 CONTROL CARD NO. 3
    1        0.     0.      0. ]
    2        0.    -1.      0. ]                           POINT COORDINATE CARDS (1 ON CARD 3
    3      1 -1.   -1.      0  ]                            INDICATES POINT 3 ON LINK 1)
    2      3  1.            1                              MEMBER PROPERTY CARD, LAST 1 CONTROL NC
    1      1      1      1 ]
    2      0      0      1 ]                               CONSTRAINT CARDS
    3      1      1      1 ]
    5      5     .05                                       MODIFICATION TO STIFFNESS MATRIX
    1        0.     0.      0.      0.      0.      0. ]
    2        19.32  19.32   19.32   0.      0.      0. ]   WEIGHT CARDS
    3        0.     0.      0.      0.      0.      0. ]
0.0 ]                                                      NORMAL MODE CARDS (2 UNRESTRAINED
1.0 ]                                                       DEGREES OF FREEDOM)
    2      1      1      1      1      1      0 ]
    0                                           ]          LINK CARDS (LINK MOTION COMPLET
           0      0                           1 ]           DEFINED AT POINT 2 WHICH IS ON
.15916                                                     NATURAL FREQUENCY CARD
0.     0.     0.     0.     0.     0.     0.     .010      MODAL DAMPING CARD
```

NOTE NO MEMBER CARDS, SINCE NO MEMBERS

SYSTEM PROCESSING INPUT (SEPARATE RUN AT SAME TIME)

```
         1      0 ----------------------------------------------- CONTROL CARD NO. 1
         2      2      7 ---------------------------------------- CONTROL CARD NO. 4
         8      9     10      11      12      13      14 -------- MODES TO BE DELETED
         1      2 ----------------------------------------------- RECORD NO. ON LOGICAL TAPE 11
         1      2 ----------------------------------------------- BASIC SYSTEM NO. OF EACH SYSTEM
         1      0      0 ---------------------------------------- CONTROL CARD NO. 5
         1      1      2 ---------------------------------------- JOINT CARD
 .7071    -.7071  0.    .7071   .7071   0.      0.      0.    1.- SPATIAL TRANSFORMATION SYSTEM 1 TO COMMON
 -.7071   -.7071  0.    .7071  -.7071   0.      0.      0.    1.- SPATIAL TRANSFORMATION SYSTEM 2 TO COMMON
         2      1      1       1       1       1       0      0 ]
         3      3      1       1       1       0       0      0 ]- COMPATIBILITY CARDS
         2      0      1 ---------------------------------------- CONTROL CARD NO. 6
         2      2 ----------------------------------------------- CONTROL CARD NO. 7
```

FORCED RESPONSE INPUT

```
         3      0 ----------------------------------------------- CONTROL CARD NO. 1
         1      1      1 ---------------------------------------- CONTROL CARD NO. 8
         1 ------------------------------------------------------ SYSTEM NO. AT CRITICAL POINT
         7 ------------------------------------------------------ DEGREE OF FREEDOM AT CRITICAL POINT
 0.7        1.2     .005 ---------------------------------------- FREQUENCY INFORMATION
         0      0 1.0  0.      0.      0.      0.      0.      2 - LOAD INFORMATION (1-g SINUSOIDAL
                                                                   ACCELERATION AT BASE ALONG X3 AXIS)
```

POINT ACCELERATIONS AND MEMBER LOAD INPUT (SEPARATE RUN)

```
         4      0 ----------------------------------------------- CONTROL CARD NO. 1
         2      1      1       1 -------------------------------- CONTROL CARD NO. 9
         1 ------------------------------------------------------ LOAD TYPE
 0.90500E 00 0.10000E 01 0.           0.          0.          0.]
 0.          0.33762E 01 0.94559E 01                             ]
 0.90500E 00 0.          0.           0.          0.          0.]- PARTICIPATION FACTORS
 0.         -0.85573E 01 0.29260E 02                             ]
         1 ------------------------------------------------------ NO. OF LOADS ]
         1      0      1 ---------------------------------------- OPTION CARD  ]- SYSTEM 1
         1 ------------------------------------------------------ NO. OF LOADS ]
         1      0      1 ---------------------------------------- OPTION CARD  ]- SYSTEM 2
```

### *B. Output Data*

The following pages are facsimiles of printout output data.

BASIC SYSTEM PRIME

| STRUCTURE TYPE | TAPE POSITION | NO. OF LINKS | NORMAL MODES | DAMP CODE | ALTER CODE | NO. OF JOINTS | NO. OF MEMBERS | ATTACH MODES |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 | -0 | 3 | 0 | 1 |

| MASS CODE | JOINTS RESTRAINED | NO. OF JOINT | OUT CODE | EDIT CODE | RIGID BODY CODE | ELASTIC MODULUS | POISSON'S RATIO |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 6 | 0 | 1 | 1 | 0. | C. |

JOINT COORDINATES

| JOINT | LINK | X1 | X2 | X3 |
|---|---|---|---|---|
| 1 | -0 | 0. | C. | 0. |
| 2 | -0 | 0.100CE 01 | C. | 0. |
| 3 | -0 | 0.200CE 01 | -C.1CC0E 01 | 0. |

RESTRAINTS

| JOINT | I1 | I2 | I3 | I4 | I5 | I6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |

STIFFNESS MATRIX ALTERATIONS

| I | J | INCREMENT |
|---|---|---|
| 7 | 7 | C.10C0E C1 |

WEIGHT MATRIX

| JOINT | W1 | W2 | W3 | W4 | W5 | W6 |
|---|---|---|---|---|---|---|
| 1 | C. | 0. | 0. | 0. | 0. | 0. |
| 2 | C.38640E 03 | 0.38640E 03 | 0.38640E 03 | 0. | 0. | 0. |
| 3 | C. | 0. | 0. | 0. | 0. | 0. |

ATTACHMENT MODE LOADS

| JOINT | L1 | L2 | L3 | L4 | L5 | L6 |
|---|---|---|---|---|---|---|
| 2 | C.10000E C1 | 0. | 0. | -0. | -0. | -0. |

MODAL VECTOR MATRIX

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| JOINT 1 | | | | | | | |
| 1 | 0.1000CE 01 | 0. | 0. | C. | 0. | -C. | C. |
| 2 | 0. | C.1C00CE C1 | 0. | -0. | 0. | 0. | 0. |
| 3 | 0. | 0. | C.1C000E 01 | 0. | -0. | 0. | 0. |
| 4 | 0. | 0. | 0. | 0.10000E 01 | 0. | C. | C. |
| 5 | 0. | 0. | 0. | C. | 0.1000CE 01 | 0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | C.1C000E 01 | C. |
| JOINT 2 | | | | | | | |
| 1 | 0.1000CE 01 | 0. | 0. | 0. | 0. | -0. | 0.10000E 01 |
| 2 | 0. | 0.10000E C1 | 0. | -C. | 0. | 0.1C0C0E C1 | C. |
| 3 | C. | 0. | 0.1C000E 01 | 0. | -0.1000CE 01 | 0. | 0. |
| 4 | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | 0. |
| 5 | 0. | 0. | 0. | C. | 0.1C00CE 01 | 0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | C.1C0C0E C1 | 0. |
| JOINT 3 | | | | | | | |
| 1 | 0.1000CE 01 | 0. | 0. | 0. | 0. | 0.10000E 01 | C. |
| 2 | 0. | 0.10000E 01 | 0. | -0. | 0. | 0.2C000E C1 | 0. |
| 3 | 0. | C. | 0.1C000E 01 | -0.10000E 01 | -0.2000CE 01 | 0. | 0. |
| 4 | 0. | 0. | C. | 0.10000E 01 | 0. | 0. | 0. |
| 5 | 0. | 0. | 0. | 0. | 0.1000CE 01 | 0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | 0.1CCC0E 01 | 0. |

GENERALIZED MASS MATRIX

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ROW 1 | 0.1000E C1 | 0. | C. | 0. | 0. | -C. | 0.100CE 01 |
| ROW 2 | 0. | 0.1000E 01 | C. | 0. | 0. | C.1000E C1 | 0. |
| ROW 3 | 0. | 0. | C.1000E 01 | C. | -C.10C0E 01 | -C. | 0. |
| ROW 4 | 0. | 0. | C. | 0. | C. | -C. | 0. |
| ROW 5 | 0. | 0. | -C.1C00E 01 | 0. | C.10C0E 01 | -C. | C. |
| ROW 6 | -0. | 0.1000E 01 | -C. | -0. | -0. | C.1000E C1 | -0. |
| ROW 7 | 0.1000E C1 | 0. | C. | 0. | C. | -C. | 0.100CE C1 |

GENERALIZED STIFFNESS MATRIX

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ROW 1 | 0. | C. | 0. | 0. | 0. | C. | 0. |
| ROW 2 | 0. | C. | C. | 0. | C. | C. | C. |
| ROW 3 | 0. | 0. | C. | 0. | C. | C. | 0. |
| ROW 4 | 0. | 0. | C. | 0. | C. | C. | C. |
| ROW 5 | 0. | 0. | 0. | 0. | C. | C. | 0. |
| ROW 6 | 0. | 0. | C. | 0. | C. | C. | 0. |
| ROW 7 | 0. | C. | C. | 0. | 0. | C. | 0.100GE 01 |

BASIC SYSTEM DAMP

| STRUCTURE TYPE | TAPE POSITION | NO. OF LINKS | NORMAL MODES | DAMP CODE | ALTER CODE | NO. OF JOINTS | NO. OF MEMBERS | ATTACH MODES |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 1 | -0 | 3 | 1 | 0 |

| MASS CODE | JOINTS RESTRAINED | NO. OF JOINT | OUT CODE | EDIT CODE | RIGID BODY CODE | ELASTIC MODULUS | POISSON'S RATIO |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 1 | 1 | 1 | 1.0000E 06 | 0. |

JOINT COORDINATES

| JOINT | LINK | X1 | X2 | X3 |
|---|---|---|---|---|
| 1 | -0 | 0. | 0. | 0. |
| 2 | -0 | 0. | -0.1000E 01 | 0. |
| 3 | 1 | -0.1000E 01 | -0.1000E 01 | 0. |

MEMBER PROPERTIES

| JOINT A | JOINT B | A1 | A2 | A3 |
|---|---|---|---|---|
| 2 | 3 | 0.1000E 01 | -0. | 0.1000E 06 |

RESTRAINTS

| JOINT | I1 | I2 | I3 | I4 | I5 | I6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | | | |
| 2 | 0 | 0 | 1 | | | |
| 3 | 1 | 1 | 1 | | | |

STIFFNESS MATRIX ALTERATIONS

| I | J | INCREMENT |
|---|---|---|
| 5 | 5 | 0.5000E-01 |

STIFFNESS MATRIX

```
ROW    1
 0.1000E 07  0.
ROW    2
 0.          0.5000E-01
```

WEIGHT MATRIX

| JOINT | W1 | W2 | W3 | W4 | W5 | W6 |
|---|---|---|---|---|---|---|
| 1 | 0. | 0. | 0. | 0. | 0. | 0. |
| 2 | 0.19320E 02 | 0.19320E 02 | 0.19320E 02 | 0. | 0. | 0. |
| 3 | 0. | 0. | 0. | 0. | 0. | 0. |

RIGID BODY DATA FOR LINKS

| JOINT | I1 | I2 | I3 | I4 | I5 | I6 |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | -0 | -0 | -0 | -0 | -0 | -0 |
| -0 | 0 | 0 | 0 | 0 | 0 | 1 |

MODAL VECTOR MATRIX

JOINT 1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.10000E 01 | 0. | 0. | 0. | 0. | -0. | 0. | 0. |
| 2 | 0. | 0.10000E 01 | 0. | -0. | 0. | 0. | 0. | 0. |
| 3 | 0. | 0. | 0.10000E 01 | 0. | -0. | 0. | 0. | 0. |
| 4 | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | 0. | 0. |
| 5 | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. |

JOINT 2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.10000E 01 | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. |
| 2 | 0. | 0.10000E 01 | 0. | -0. | 0. | 0. | 0. | 0.10000E 01 |
| 3 | 0. | 0. | 0.10000E 01 | -0.10000E 01 | -0. | 0. | 0. | 0. |
| 4 | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | 0. | 0. |
| 5 | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. |

JOINT 3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.10000E 01 | 0. | 0. | 0. | 0. | 0.10000E 01 | -0. | 0. |
| 2 | 0. | 0.10000E 01 | 0. | -0. | 0. | -0.10000E 01 | -0.10000E 01 | 0. |
| 3 | 0. | 0. | 0.10000E 01 | -0.10000E 01 | 0.10000E 01 | 0. | -0. | 0. |
| 4 | 0. | 0. | 0. | 0.10000E 01 | 0. | 0. | -0. | 0. |
| 5 | 0. | 0. | 0. | 0. | 0.10000E 01 | 0. | -0. | 0. |
| 6 | 0. | 0. | 0. | 0. | 0. | 0.10000E 01 | 0.10000E 01 | 0. |

GENERALIZED MASS MATRIX

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ROW 1 | 0.5000E-01 | 0. | 0. | -0. | 0. | 0.5000E-01 | 0. | 0. |
| ROW 2 | 0. | 0.5000E-01 | 0. | -0. | 0. | 0. | 0. | 0.5000E-01 |
| ROW 3 | 0. | 0. | 0.5000E-01 | -0.5000E-01 | 0. | 0. | 0. | 0. |
| ROW 4 | -0. | -0. | -0.5000E-01 | 0.5000E-01 | -0. | -0. | -0. | -0. |
| ROW 5 | 0. | 0. | 0. | -0. | 0. | 0. | 0. | 0. |
| ROW 6 | 0.5000E-01 | 0. | 0. | -0. | 0. | 0.5000E-01 | 0. | 0. |
| ROW 7 | 0. | 0. | 0. | -0. | 0. | 0. | 0. | 0. |
| ROW 8 | 0. | 0.5000E-01 | 0. | -0. | 0. | 0. | 0. | 0.5000E-01 |

GENERALIZED STIFFNESS MATRIX

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ROW 1 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 2 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 3 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 4 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 5 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 6 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 7 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0. |
| ROW 8 | 0. | 0. | 0. | 0. | 0. | 0. | 0. | 0.5000E-01 |

GENERALIZED DAMPING MATRIX DIAGONAL

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0. | 0. | 0. | 0. | 0. | 0. | 0. | 1.0000E-02 |

```
                                   SYSTEM PROCESSING

                              NUMBER OF BASIC SYSTEMS =   2

                                NUMBER OF SYSTEMS =   2

                               NUMBER OF DELETIONS =   7

                              DELETED DEGREES OF FREEDOM

     8         9         10         11         12         13         14

                                  BASIC SYSTEMS USED

     1         2

                          SYSTEM-BASIC SYSTEM CORRESPONDENCE

                                 SYSTEM - BASIC SYSTEM

                                    1          1
                                    2          2

SYSTEM  2 IS ATTACHED TO  1 SYSTEMS

ATTACHMENT  1 IS TO SYSTEM  1 USING  1 TRANSFORMATIONS.

                                       TRANSFORMATION SET  1

          TRANSFORMATION FOR SYSTEM  1                     TRANSFORMATION FOR SYSTEM  2

        0.70710E 00-0.70710E 00 0.                        -0.7071CE 00-0.70710E 00 0.
        0.70710E 00 0.7071CE 00 0.                         0.7071CE 00-0.70710E 00 0.
        0.          0.          0.100C0E 01                0.          0.          0.10000E 01

        JCINT IN            JOINT IN           COMPATIBILITY
        SYSTEM  1           SYSTEM  2
            2                   1              1  1  1  1  0  0
            3                   3              1  1  1  0  C  0


                    SYSTEM TRANSFORMATION FOR SYSTEM 1

ROW   1
 0.1000E 01  C.          C.          0.          0.          0.          0.          0.
ROW   2
 0.          0.1C00E 01  0.          0.          0.          0.          0.          0.
ROW   3
 0.          0.          C.1000E 01  0.          0.          C.          0.          0.
ROW   4
 0.          0.          C.          0.1000E 01  0.          C.          0.          0.
ROW   5
 0.          0.          0.          0.          0.10C0E 01  C.          0.          0.
ROW   6
 0.          0.          0.          0.          0.          0.1000E C1  0.          0.
ROW   7
 0.          0.          0.          0.          0.          C.          0.1000E 01  0.


                    SYSTEM TRANSFORMATION FOR SYSTEM 2

ROW   1
-0.3725E-08  1.0000E 00 -0.          0.          C.          1.0000E C0 -0.3725E-08  0.
ROW   2
-1.0000E 00 -0.3725E-08  C.         -0.         -0.         -C.3725E-C8 -1.0000E 00 -0.
ROW   3
-0.         -0.          0.1000E 01  0.         -C.10C0E 01 -C.         -0.          0.
ROW   4
-0.         -0.         -0.          0.3725E-08  1.00C0E 00 -C.         -0.          0.
ROW   5
 0.          0.          C.         -1.00C0E 00 -C.7451E-08  C.          0.         -0.
ROW   6
 0.          0.          0.         -0.         -0.          1.0000E C0  0.         -0.
ROW   7
 0.7451E-08 -0.7451E-08 -0.          C.          C.         -0.7451E-C8 -1.000CE 00  0.
ROW   8
 0.          0.          0.          0.          C.          C.          0.          0.10C0E 01
```

```
                                                GENERALI7ED MASS MATRIX
ROW   1
 0.1050E C1 -0.1735E-17  C.          0.          C.          -C.1863E-C9  C.105CE 01 -0.5000E-01
ROW   2
-0.1735E-17  0.1050E 01  C.          C.          C.           C.1100E 01 -0.1735E-17 -C.1863E-09
ROW   3
 0.          0.          C.1050E 01 -0.1863E-09 -C.11C0E 01 -C.          -C.          0.
ROW   4
 0.          0.         -C.1863E-09  0.6939E-18  C.3725E-09 -C.          -C.          0.
ROW   5
 0.          0.         -C.1100E 01  0.3725E-09  C.12C0E 01 -C.          -0.          0.
ROW   6
-0.1863E-C9  0.1100E C1 -0.         -0.         -C.           C.1200E C1 -C.1863E-C9 -0.1663E-C9
ROW   7
 0.1050E 01 -0.1735E-17 -C.         -0.         -C.          -C.1863E-C9  0.1050E 01 -0.5000E-01
ROW   8
-0.5000E-C1 -0.1863E-09  C.          C.          0.          -C.1863E-C9 -C.500CE-01  0.5000E-01



                                                GENERALIZED STIFFNESS MATRIX
ROW   1
 0.          0.          C.          0.          0.          C.          0.          0.
ROW   2
 0.          0.          C.          C.          C.          C.          0.          0.
ROW   3
 0.          C.          C.          0.          C.          C.          C.          0.
ROW   4
 0.          0.          C.          0.          C.          C.          C.          0.
ROW   5
 0.          0.          0.          0.          0.          C.          0.          0.
ROW   6
 0.          0.          C.          C.          C.          C.          C.          0.
ROW   7
 0.          0.          C.          C.          0.          C.          C.100CE 01  0.
ROW   8
 0.          C.          C.          C.          C.          C.          0.          0.5000E-01



                                                GENERALIZED DAMPING MATRIX
ROW   1
 0.          0.          C.          0.          0.          C.          0.          0.
ROW   2
 0.          0.          C.          C.          C.          C.          C.          0.
ROW   3
 0.          0.          C.          0.          0.          C.          0.          0.
ROW   4
 0.          0.          C.          0.          C.          C.          0.          0.
ROW   5
 0.          C.          0.          0.          C.          C.          C.          0.
ROW   6
 0.          C.          C.          0.          0.          C.          0.          0.
ROW   7
 0.          0.          C.          C.          C.          C.          0.          0.
ROW   8
 0.          0.          C.          0.          0.          0.          0.          1.0000E-02
CALCULATICNS ARE FOR TYPE 1 LOADING.



                                                UNDAMPED EIGENVALLES AND EIGENVECTCRS

EIGENVALUE NO.  1 =  0.80CCE 00
ASSOCIATEC EIGENVECTOR
 0.8133E 00 -0.5819E 00

EIGENVALUE NO.  2 =  0.125CE 01
ASSOCIATEC EIGENVECTOR
 0.5819E 0C  0.8133E 00



NEW REDUCTICN MATRIX, TR*F(INVERSE)*V(UNDAMPED)
ROW  1
-0.          0.
ROW  2
-0.          C.
ROW  3
-0.          0.
ROW  4
-0.          0.
ROW  5
-0.          0.
ROW  6
-0.          0.
ROW  7
 0.6667E C0  0.7453E 00
ROW  8
-0.2666E C1  0.3727E 01
```

```
            COMPOSITE MODAL MATRIX FOR SYSTEM  1
JOINT   1
1  0.              0.
2  0.              0.
3  0.              0.
4  0.              0.
5  0.              0.
6  0.              0.
JOINT   2
1  0.66672E 00   0.74531E 00
2  0.              0.
3  0.              0.
4  0.              0.
5  0.              0.
6  0.              0.
JOINT   3
1  0.              0.
2  0.              0.
3  0.              0.
4  0.              0.
5  0.              0.
6  0.              0.

            COMPOSITE MODAL MATRIX FOR SYSTEM  2
JOINT   1
1 -0.24837E-08 -0.27765E-08
2 -0.66672E 00 -0.74531E 00
3 -0.             0.
4 -0.             0.
5 -0.             0.
6 -0.             0.
JOINT   2
1 -0.24837E-08 -0.27765E-08
2 -0.33331E 01  0.29817E 01
3 -0.             0.
4 -0.             0.
5 -0.             0.
6 -0.             0.
JOINT   3
1 -0.24837E-08 -0.27765E-08
2 -0.             0.
3 -0.             0.
4 -0.             0.
5 -0.             0.
6 -0.66672E 00 -0.74531E 00


                                                 TRANSFORMED DAMPING MATRIX
ROW   1
 0.7110E-01 -0.9938E-01
ROW   2
-0.9938E-01  0.1389E-00


   EIGENVALUE NO.  1
 -0.348509E-01   -0.904080E 00
   ASSOCIATED EIGENVECTOR
  0.100000E 01    0.                  0.219939E-01   -0.206192E-00

   EIGENVALUE NO.  2
 -0.348509E-01    0.904080E 00
   ASSOCIATED EIGENVECTOR
  0.100000E 01   -0.                  0.219939E-01    0.206192E-00

   EIGENVALUE NO.  3
 -0.701491E-01   -0.110308E 01
   ASSOCIATED EIGENVECTOR
 -0.304153E-01    0.257412E-00        0.100000E 01    0.

   EIGENVALUE NO.  4
 -0.701491E-01    0.110308E 01
   ASSOCIATED EIGENVECTOR
 -0.304153E-01   -0.257412E-00        0.100000E 01   -0.
```

```
        ORTHOGONALITY CHECK
     MAXIMUM MODULUS OF OFF DIAGONAL ELEMENTS NORMALIZED BY DIAGONAL ELEMENTS =    0.
     MAXIMUM MODULUS OF DIAGONAL ELEMENTS =    0.211225E 01
     MINIMUM MODULUS OF DIAGONAL ELEMENTS =    0.169195E 01


        COMPLEX DIAGONAL ELEMENTS OF ORTHOGONAL CHECK MATRIX
 -0.222841E-01  -0.169181E 01     -0.222841E-01   0.169181E 01      -0.253747E-01  -0.211210E 01
 -0.253747E-01   0.211210E 01


        COMPLEX REDUCTION MATRIX,TR . F(INVERSE) . V(UNDAMPED) . V(DAMPED)
COLUMN  1 REAL
 0.         0.         0.         0.         0.         0.          0.6831E 00 -0.2584E 01
        IMAGINARY
-0.        -0.        -0.        -0.        -0.        -0.         -0.1537E-00 -0.7685E 00
COLUMN  2 REAL
 0.         0.         0.         0.         0.         0.          0.6831E 00 -0.2584E 01
        IMAGINARY
 0.         0.         0.         0.         0.         0.          0.1537E-00  0.7685E 00
COLUMN  3 REAL
 0.         0.         0.         0.         0.         0.          0.7250E 00  0.3808E 01
        IMAGINARY
 0.         0.         0.         0.         0.         0.          0.1716E-00 -0.6864E 00
COLUMN  4 REAL
 0.         0.         0.         0.         0.         0.          0.7250E 00  0.3808E 01
        IMAGINARY
-0.        -0.        -0.        -0.        -0.        -0.         -0.1716E-00  0.6864E 00
```

CRITICAL POINTS FOR TYPE 1 LOADING

| POINT | SYSTEM | D.F. |
|---|---|---|
| 1 | 1 | 7 |

LOAD COMPONENTS FOR LOAD 1

| SYSTEM LOADED | JOINT LOADED | X1 | X2 | X3 | X4 | X5 | X6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.1000E 01 | 0. | 0. | 0. | 0. | 0. |

COMPLEX LOAD VECTOR ON

| | |
|---|---|
| -0.84649E 00 | 0.12294E-00 |
| -0.84649E 00 | -0.12294E-00 |
| -0.57088E 00 | -0.21452E-00 |
| -0.57088E 00 | 0.21452E-00 |

RESPONSE FOR CRITICAL POINT 1 LOAD 1

| FREQUENCY | RESPONSE | | MAGNITUDE |
|---|---|---|---|
| 0.7000E 00 | 0.2157E 01 | -0.2795E-01 | 0.2157E 01 |
| 0.7050E 00 | 0.2196E 01 | -0.3077E-01 | 0.2196E 01 |
| 0.7100E 00 | 0.2237E 01 | -0.3392E-01 | 0.2237E 01 |
| 0.7150E 00 | 0.2279E 01 | -0.3744E-01 | 0.2280E 01 |
| 0.7200E 00 | 0.2324E 01 | -0.4138E-01 | 0.2325E 01 |
| 0.7250E 00 | 0.2372E 01 | -0.4580E-01 | 0.2372E 01 |
| 0.7300E 00 | 0.2422E 01 | -0.5077E-01 | 0.2422E 01 |
| 0.7350E 00 | 0.2474E 01 | -0.5637E-01 | 0.2475E 01 |
| 0.7400E 00 | 0.2530E 01 | -0.6269E-01 | 0.2530E 01 |
| 0.7450E 00 | 0.2588E 01 | -0.6984E-01 | 0.2589E 01 |
| 0.7500E 00 | 0.2650E 01 | -0.7796E-01 | 0.2651E 01 |
| 0.7550E 00 | 0.2716E 01 | -0.8719E-01 | 0.2717E 01 |
| 0.7600E 00 | 0.2785E 01 | -0.9772E-01 | 0.2787E 01 |
| 0.7650E 00 | 0.2860E 01 | -0.1098E-00 | 0.2862E 01 |
| 0.7700E 00 | 0.2939E 01 | -0.1236E-00 | 0.2941E 01 |
| 0.7750E 00 | 0.3023E 01 | -0.1395E-00 | 0.3026E 01 |
| 0.7800E 00 | 0.3113E 01 | -0.1578E-00 | 0.3117E 01 |
| 0.7850E 00 | 0.3210E 01 | -0.1791E-00 | 0.3215E 01 |
| 0.7900E 00 | 0.3313E 01 | -0.2039E-00 | 0.3320E 01 |
| 0.7950E 00 | 0.3425E 01 | -0.2328E-00 | 0.3433E 01 |
| 0.8000E 00 | 0.3545E 01 | -0.2668E-00 | 0.3555E 01 |
| 0.8050E 00 | 0.3675E 01 | -0.3067E-00 | 0.3687E 01 |
| 0.8100E 00 | 0.3815E 01 | -0.3541E-00 | 0.3831E 01 |
| 0.8150E 00 | 0.3967E 01 | -0.4104E-00 | 0.3988E 01 |
| 0.8200E 00 | 0.4132E 01 | -0.4778E-00 | 0.4160E 01 |
| 0.8250E 00 | 0.4311E 01 | -0.5588E 00 | 0.4348E 01 |
| 0.8300E 00 | 0.4506E 01 | -0.6568E 00 | 0.4554E 01 |
| 0.8350E 00 | 0.4718E 01 | -0.7759E 00 | 0.4781E 01 |
| 0.8400E 00 | 0.4948E 01 | -0.9216E 00 | 0.5033E 01 |
| 0.8450E 00 | 0.5195E 01 | -0.1101E 01 | 0.5311E 01 |
| 0.8500E 00 | 0.5460E 01 | -0.1322E 01 | 0.5618E 01 |
| 0.8550E 00 | 0.5741E 01 | -0.1596E 01 | 0.5959E 01 |
| 0.8600E 00 | 0.6031E 01 | -0.1938E 01 | 0.6334E 01 |
| 0.8650E 00 | 0.6320E 01 | -0.2361E 01 | 0.6747E 01 |
| 0.8700E 00 | 0.6590E 01 | -0.2885E 01 | 0.7193E 01 |
| 0.8750E 00 | 0.6811E 01 | -0.3524E 01 | 0.7669E 01 |
| 0.8800E 00 | 0.6940E 01 | -0.4289E 01 | 0.8159E 01 |
| 0.8850E 00 | 0.6920E 01 | -0.5171E 01 | 0.8639E 01 |
| 0.8900E 00 | 0.6685E 01 | -0.6133E 01 | 0.9072E 01 |
| 0.8950E 00 | 0.6180E 01 | -0.7094E 01 | 0.9409E 01 |
| 0.9000E 00 | 0.5393E 01 | -0.7943E 01 | 0.9601E 01 |
| 0.9050E 00 | 0.4376E 01 | -0.8557E 01 | 0.9611E 01 |
| 0.9100E 00 | 0.3244E 01 | -0.8859E 01 | 0.9434E 01 |
| 0.9150E 00 | 0.2134E 01 | -0.8841E 01 | 0.9095E 01 |
| 0.9200E 00 | 0.1158E 01 | -0.8563E 01 | 0.8641E 01 |
| 0.9250E 00 | 0.3710E-00 | -0.8118E 01 | 0.8126E 01 |
| 0.9300E 00 | -0.2177E-00 | -0.7593E 01 | 0.7596E 01 |
| 0.9350E 00 | -0.6311E 00 | -0.7053E 01 | 0.7081E 01 |
| 0.9400E 00 | -0.9037E 00 | -0.6539E 01 | 0.6601E 01 |
| 0.9450E 00 | -0.1070E 01 | -0.6072E 01 | 0.6165E 01 |

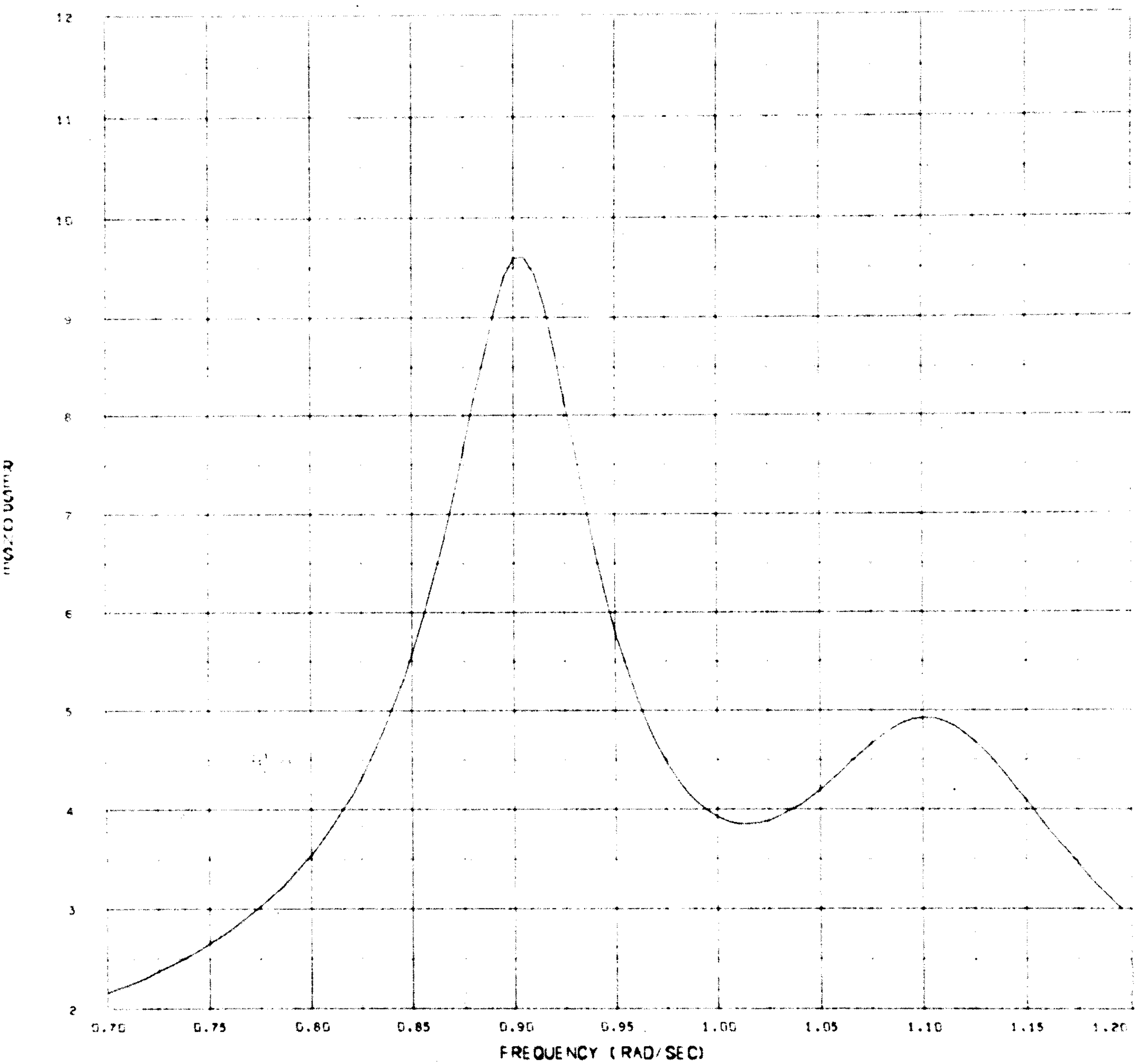
RESPONSE FOR CRITICAL POINT 1 LOAD 1

| FREQUENCY | RESPONSE | | MAGNITUDE |
|---|---|---|---|
| 0.9500E 00 | -0.1158E 01 | -0.5658E 01 | 0.5776E 01 |
| 0.9550E 00 | -0.1192E 01 | -0.5300E 01 | 0.5432E 01 |
| 0.9600E 00 | -0.1189E 01 | -0.4993E 01 | 0.5133E 01 |
| 0.9650E 00 | -0.1161E 01 | -0.4733E 01 | 0.4873E 01 |
| 0.9700E 00 | -0.1116E 01 | -0.4514E 01 | 0.4650E 01 |
| 0.9750E 00 | -0.1062E 01 | -0.4333E 01 | 0.4461E 01 |
| 0.9800E 00 | -0.1002E 01 | -0.4184E 01 | 0.4302E 01 |
| 0.9850E 00 | -0.9414E 00 | -0.4063E 01 | 0.4171E 01 |
| 0.9900E 00 | -0.8811E 00 | -0.3969E 01 | 0.4065E 01 |
| 0.9950E 00 | -0.8236E 00 | -0.3897E 01 | 0.3983E 01 |
| 1.0000E 00 | -0.7703E 00 | -0.3846E 01 | 0.3922E 01 |
| 0.1005E 01 | -0.7227E 00 | -0.3813E 01 | 0.3881E 01 |
| 0.1010E 01 | -0.6820E 00 | -0.3798E 01 | 0.3859E 01 |
| 0.1015E 01 | -0.6495E 00 | -0.3799E 01 | 0.3854E 01 |
| 0.1020E 01 | -0.6264E 00 | -0.3814E 01 | 0.3865E 01 |
| 0.1025E 01 | -0.6141E 00 | -0.3842E 01 | 0.3891E 01 |
| 0.1030E 01 | -0.6141E 00 | -0.3883E 01 | 0.3931E 01 |
| 0.1035E 01 | -0.6279E 00 | -0.3934E 01 | 0.3984E 01 |
| 0.1040E 01 | -0.6575E 00 | -0.3994E 01 | 0.4048E 01 |
| 0.1045E 01 | -0.7045E 00 | -0.4062E 01 | 0.4122E 01 |
| 0.1050E 01 | -0.7707E 00 | -0.4134E 01 | 0.4205E 01 |
| 0.1055E 01 | -0.8580E 00 | -0.4208E 01 | 0.4294E 01 |
| 0.1060E 01 | -0.9677E 00 | -0.4280E 01 | 0.4388E 01 |
| 0.1065E 01 | -0.1101E 01 | -0.4346E 01 | 0.4484E 01 |
| 0.1070E 01 | -0.1257E 01 | -0.4402E 01 | 0.4578E 01 |
| 0.1075E 01 | -0.1436E 01 | -0.4441E 01 | 0.4668E 01 |
| 0.1080E 01 | -0.1635E 01 | -0.4459E 01 | 0.4749E 01 |
| 0.1085E 01 | -0.1850E 01 | -0.4450E 01 | 0.4819E 01 |
| 0.1090E 01 | -0.2076E 01 | -0.4410E 01 | 0.4874E 01 |
| 0.1095E 01 | -0.2305E 01 | -0.4335E 01 | 0.4910E 01 |
| 0.1100E 01 | -0.2531E 01 | -0.4226E 01 | 0.4926E 01 |
| 0.1105E 01 | -0.2745E 01 | -0.4083E 01 | 0.4920E 01 |
| 0.1110E 01 | -0.2940E 01 | -0.3908E 01 | 0.4891E 01 |
| 0.1115E 01 | -0.3110E 01 | -0.3709E 01 | 0.4840E 01 |
| 0.1120E 01 | -0.3251E 01 | -0.3490E 01 | 0.4769E 01 |
| 0.1125E 01 | -0.3360E 01 | -0.3259E 01 | 0.4681E 01 |
| 0.1130E 01 | -0.3438E 01 | -0.3023E 01 | 0.4578E 01 |
| 0.1135E 01 | -0.3485E 01 | -0.2788E 01 | 0.4463E 01 |
| 0.1140E 01 | -0.3505E 01 | -0.2560E 01 | 0.4340E 01 |
| 0.1145E 01 | -0.3501E 01 | -0.2341E 01 | 0.4212E 01 |
| 0.1150E 01 | -0.3477E 01 | -0.2135E 01 | 0.4080E 01 |
| 0.1155E 01 | -0.3437E 01 | -0.1943E 01 | 0.3948E 01 |
| 0.1160E 01 | -0.3383E 01 | -0.1766E 01 | 0.3816E 01 |
| 0.1165E 01 | -0.3320E 01 | -0.1604E 01 | 0.3687E 01 |
| 0.1170E 01 | -0.3249E 01 | -0.1457E 01 | 0.3561E 01 |
| 0.1175E 01 | -0.3174E 01 | -0.1324E 01 | 0.3439E 01 |
| 0.1180E 01 | -0.3096E 01 | -0.1204E 01 | 0.3322E 01 |
| 0.1185E 01 | -0.3016E 01 | -0.1096E 01 | 0.3209E 01 |
| 0.1190E 01 | -0.2935E 01 | -0.9987E 00 | 0.3101E 01 |
| 0.1195E 01 | -0.2856E 01 | -0.9113E 00 | 0.2998E 01 |

PARTICIPATION FACTORS FOR MAXIMUM RESPONSE AT W = 0.9050

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.10000E 01 | 0. | 0. | 0. | 0. | 0. | 0.33762E 01 | 0.94556E 01 |
| 0. | 0. | 0. | 0. | 0. | 0. | -0.85573E 01 | 0.29260E 02 |

RESPONSE PLOT COMPLETED.


12
11
10
9
8
7
6
5
4
3
2
0.70
0.75
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20
FREQUENCY (RAD/SEC)

SYSTEM NUMBER 1
LOACING CONCITICN NUMBER 1
FREQUENCY = 0.9050

ATTACHMENT MODE LOAD MULTIPLIERS

| REAL | IMAGINARY |
|---|---|
| 2.5398E-01 | 1.8909E 00 |

ACCELERATION VECTORS FOR SYSTEM 1

| | MODAL DEFLECTION REAL | MODAL DEFLECTION IMAGINARY | MASS ACCELERATION REAL | MASS ACCELERATION IMAGINARY |
|---|---|---|---|---|
| JOINT 1 | | | | |
| 1 | 1.0000E 00 | -C. | 1.000CE 00 | 0. |
| 2 | C. | -C. | -0. | 0. |
| 3 | C. | -C. | -0. | 0. |
| 4 | C. | -C. | -0. | 0. |
| 5 | C. | -C. | -0. | 0. |
| 6 | C. | -C. | -0. | 0. |
| JOINT 2 | | | | |
| 1 | 4.3762E 00 | -8.5573E 00 | 4.3762E 00 | -8.5573E C0 |
| 2 | C. | -C. | -0. | 0. |
| 3 | C. | -C. | -0. | 0. |
| 4 | C. | -C. | -0. | 0. |
| 5 | C. | -C. | -0. | 0. |
| 6 | C. | -C. | -0. | 0. |
| JOINT 3 | | | | |
| 1 | 1.0C00E 00 | -C. | 1.000CE 00 | 0. |
| 2 | C. | -C. | -0. | 0. |
| 3 | C. | -C. | -0. | 0. |
| 4 | C. | -C. | -0. | 0. |
| 5 | C. | -C. | -0. | 0. |
| 6 | C. | -C. | -0. | 0. |

PRINT-OUT FOR SYSTEM 1 COMPLETED.

SYSTEM NUMBER 2
LOACING CONCITICN NUMBER 1
FREQUENCY = 0.905C

ACCELERATION VECTORS FOR SYSTEM 2

| | MODAL DEFLECTION REAL | MODAL DEFLECTION IMAGINARY | MASS ACCELERATICN REAL | MASS ACCELERATICN IMAGINARY |
|---|---|---|---|---|
| JOINT 1 | | | | |
| 1 | -1.63C3E-08 | 3.1878E-08 | -1.6303E-08 | 3.1878E-C8 |
| 2 | -4.3762E 00 | 8.5573E 00 | -4.3762E 00 | 8.5573E CC |
| 3 | C. | C. | 0. | 0. |
| JOINT 2 | | | | |
| 1 | -1.6303E-08 | 3.1878E-08 | -1.6303E-08 | 3.1878E-C8 |
| 2 | 5.0797E 00 | 3.7817E 01 | -2.158CE-01 | 3.9531E C1 |
| 3 | C. | C. | 0. | 0. |
| JOINT 3 | | | | |
| 1 | -1.63C3E-08 | 3.1878E-08 | -1.6303E-08 | 3.1878E-C8 |
| 2 | -1.000OE 00 | C. | -1.000CE 00 | 0. |
| 3 | C. | C. | 0. | 0. |

MEMBER OUTPLT FOR SYSTEM 2

USING MODAL DEFLECTION DISPLACEMENTS

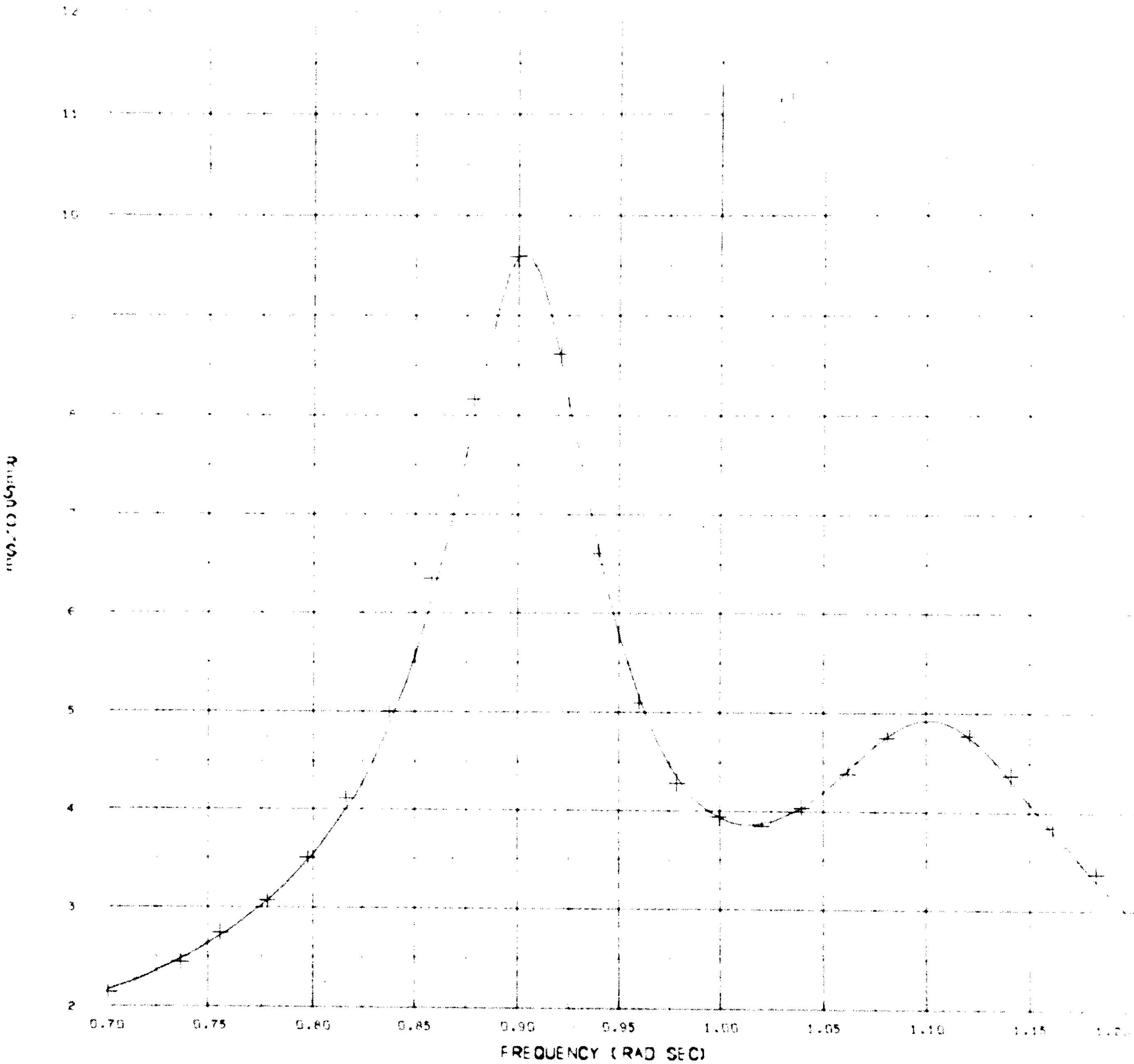
| A/I/K | GAMMA | | X1 | | THETA1 | | X2 | | THETA2 |
|---|---|---|---|---|---|---|---|---|---|
| 1.0000E 00 | -1.0000E OC | 0. | -0. | C. | -C. | 0. | -0. | 0. | -0. |
| -0. | -0. | 0. | -C. | C. | -C. | 0. | -0. | 0. | -0. |
| 0. | 0. | 0. | -0. | C. | -0. | 0. | -0. | 0. | -0. |

| JTA | JTB | P(REAL) | P(IMAGINARY) |
|---|---|---|---|
| 2 | 3 | C. | -0. |

PRINT-OUT FOR SYSTEM 2 COMPLETED.

### C. Comparison with Other Methods

The following page is a facsimile printout graph, showing data points based on an equation given in "Mechanical Vibration" by Den Hartog of the Massachusetts Institute of Technology, as cited earlier in this Report.


11
10
6
5
4
3
2
0.70
0.75
0.80
0.85
0.90
0.95
1.00
1.05
1.10
FREQUENCY (RAD SEC)

# IX. USEFUL TECHNIQUES

## A. Modifying Output

If the response output is desired for a point not along one of the coordinate axes, an additional one-point rigid system may be attached at that point with the proper orientation of coordinate axes.

## B. Using Test Data

When modal test results are being used and it is not desired to input all the test points to develop the mass matrix, the test mass matrix may be developed by using 0-point masses and modifications to the generalized mass matrix.

## C. Using Two Systems to Represent a Single Structure

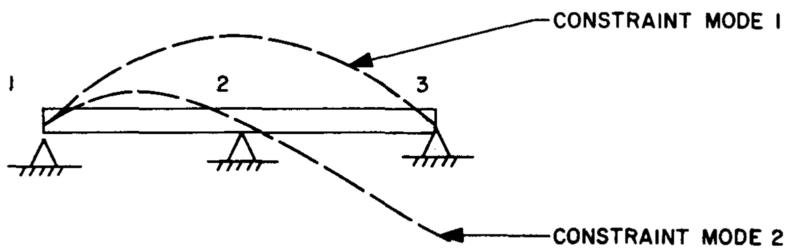
If the modes or structure are known to be subdivided into two distinct sets such that all displacements are completely defined by one system or the other, never a combination of the two (such as dividing planar structures into in-plane and out-of-plane models), the two models may be attached to each other at a point or set of points rigidly connected to one another and the combination attached to other systems only through the system defining the motion of the attachment. No mass should be associated with the points for which the motion is defined in the other model.

# X. PITFALLS

The use of the modal combination program has resulted in many undesired results and the current writeup has incorporated statements intended to prevent repetition of these errors.

When the generalized stiffness (or damping) matrix of the composite structure has large terms (larger than roundoff) in the rows and columns corresponding to the rigid body motion of the primary system, it is indicative of a discrepancy in geometry. Sometimes these discrepancies are intentional; other times they are due to errors. It is legitimate to match the motion of two points at different nearby locations if there is not going to be any rotational input of the primary system, but this will give rise to elastic deformation of the structure when the primary system is subjected to rigid body rotations. The use of rigid body modes of systems other than the primary system as independent (such use is improper) also gives rise to large elements in the rigid body rows and columns of the composite stiffness matrix.

Care must be taken that dependent modes are not chosen that have displacements at attachment points that are linear combinations of one another. A simple example is given in Sketch No. 5 and in the tabular data below for an indeterminantly supported beam.




**Sketch No. 5**

Only one vertical support motion may be used as a constraint mode, as a second constraint mode would be a linear combination of the first and rigid body modes.

| Point | Mode | | | |
|---|---|---|---|---|
| | Trans. | Rot. | Const. 1 | Const. 2 |
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 1.0 | 1.0 | 1.0 | 0 |
| 3 | 1.0 | 2.0 | 0 | 1.0 |
| 0.5 × Rot. − 0.5 × const. 1 = const. 2 | | | | |

A particular example that has caused repeated trouble is the statically indeterminate attachment of a rigid structure or a structure that is attached at a rigid part (possibly rigid only in a plane or out of a plane). All modes being eliminated in such a case can't be from the system being added in this case, as the displacements of the points being attached in one of the modes are a linear combination of the displacements in the other modes. If both the system being attached and the system to which it is attached are rigid in the area of attachment, only a statically determinate attachment can be used. If the areas of attachment aren't absolutely rigid but are quite stiff relative to the remainder of the structure, it is sometimes advantageous to idealize the area as rigid and use only a statically determinate attachment. It must be recognized that the modification of attachment locally changes the load paths, and some of the member loads in the immediate area will be in error. The error in the remainder of the structure will be small if the portion of the structure idealized as rigid is, in fact, stiff relative to the rest of the structure.

A check should always be made to ensure that a mode is removed corresponding to each compatibility condition enforced.

If a planar or linear composite structure is being analyzed, it may be necessary to input arbitrary out-of-plane masses to prevent $[M_{RR}]$ from being singular.

## ACKNOWLEDGMENTS

Work on the modal combination program for dynamic analysis of structures was initiated to implement the method presented by W. C. Hurty in JPL TR 32-530. Many fruitful discussions were held with W. C. Hurty and W. Gayman during the course of development.

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